Meghan Allerton, “Empirical Exercise: The Dynamics of Knowing”
Journal of Macrodynamic Analysis 8 (2015): 5–10

Empirical Exercise: The Dynamics of Knowing

Meghan Allerton

With this empirical exercise I have a dual focus: first, my goal is to discover a rule for getting square roots. But I go about solving this problem with a ‘methodological interest in the procedure.’ My questions, insights, efforts to understand, judge, and know are my data as I struggle to reach toward the second and primary goal: to discover the dynamics of my own wonder. So as I puzzled over the square root problem I also made a few very difficult first steps in tackling a much more complex puzzle, my wonder and its operations in me.

The exercise is a modification of a similar problem presented in Chapter Three of *Wealth of Self and Wealth of Nations*. As McShane notes, many of us are familiar with techniques or rules, for example, to find square roots. Such techniques are sometimes taught in school and may be easily memorized. “You can use the rule with ease. But our crucial question is, do you understand the rule, the *Why* of it?”[1] I have found that the only way to come to grips with this gap between memorization and understanding is through empirical investigations into my own experience, by puzzling, and by puzzling about my puzzling. This essay is intended to document my own first steps into this odd zone in the hope that it may be of some help to others who wish to make a similar effort.

I have divided the paper into three sections. The first two sections correspond to two distinct modes of wonder, which I was able to identify in the process of completing the exercise. They are named *What-ing* and *Is-ing* for the types of questions that arose as I worked towards a solution. In the third section I summarize my results.

1. What-ing

With the first sight of my puzzle came a spontaneous question: *What is the rule for getting square roots?* This what-question was an expression of my desire to *understand*, and this desire, together with the seen

[1] Philip McShane, *Wealth of Self and Wealth of Nations: Self-Axis of the Great Ascent* (Washington D.C.: University Press of America, 1975), 19.

puzzle, created a kind of inner tension that drove me in my search for an answer. My immediate thought was to try working with an example. I chose a number, which I knew was a perfect square: *how could I find the square root of 83,521?* As I looked at the problem, a solution seemed very remote indeed. I did not know how to approach it, so I simply began trying out different possibilities that came to me as I looked. These possibilities seemed to arise partly as a result of my prior knowledge and experience.

Part of this prior knowledge included some basic algebra, as well as an understanding of the meaning of *square root.* I understood that when looking for the square root of X, one is looking for the number which, when multiplied by itself, equals X. So, when presented with the problem, 'find the square root of 83,521,' I knew I was looking for a number, R, such that R•R = 83,521. I also had a basic understanding of the base 10 numeral system. I understood that, for example, 275 means two sets of 10^2 plus seven sets of 10^1 plus five sets of 10^0, where $10^0 = 1$.

Having this background understanding, and the tension of my what-wonder driving me, I somehow suddenly came up with an idea: maybe writing the number 83,521 explicitly in powers of 10 will help. I modified my image as follows:

$$83{,}521 = 8\bullet10^4 + 3\bullet10^3 + 5\bullet10^2 + 2\bullet10^1 + 1$$

With the sight of this new arrangement of data I was able to make another leap. I saw that the highest power of ten in the square (83,521) was four, and I knew that the square root must multiply by itself to give that highest power. Since $10^2\bullet10^2 = 10^4$, the highest power of ten in the square root must be two. Maybe it would also help to write the square root, R, explicitly in powers of 10, with two being the highest power. I did:

$$\mathrm{R} = x\bullet10^2 + y\bullet10^1 + z\bullet1$$

Now, to my surprise, instead of looking for *one* variable, R, I was looking for *three* variables, x, y, and z, where x, y, and z are whole numbers from 0 to 9. With this new data I was able to modify my image again. I was looking for R such that R•R = 83,521, but now I could replace R with the expression above to get:

$$83{,}521 = (x\bullet10^2 + y\bullet10 + z)(x\bullet10^2 + y\bullet10 + z)$$

By expanding I arrived at:

$$83{,}521 = x^2 10^4 + 2xy10^3 + y^2 10^2 + 2xz10^2 + 2yz10 + z^2$$

At this point, the next step was not immediately obvious to me. How could I use this equation to find the unknown variables, x, y, and z?

I was helped again by some prior mathematical experience I had with finding two or three unknowns. I knew that sometimes, if it is possible to find one variable, then that number can be used to find the next variable, and so on. I had an idea: perhaps I could find the variables in this way if I could rewrite the above equation such that the first term contained only one variable, x, the second term contained x and y, and the third term contain all three variables. The resulting equation would be loosely of the form:

$$83{,}521 = \{x\},\ \{x, y\},\ \{x, y, z\}$$

After a fair bit of messing with algebra I eventually arrived at the following:

$$83{,}521 = x^2 \bullet 10^4 + (2x \bullet 10 + y)\, y \bullet 10^2 + (2x \bullet 10^2 + 2y \bullet 10 + z)\, z$$

If my idea was correct, then my next task was to use this equation, somehow, to find x. *How could I do this? In what way could I find x without knowing y and z?* I considered the data that I had so far. I knew that:

$$83{,}521 = x^2 \bullet 10^4 + (2x \bullet 10 + y)\, y \bullet 10^2 + (2x \bullet 10^2 + 2y \bullet 10 + z)\, z$$
$$\mathrm{R} = x \bullet 10^2 + y \bullet 10^1 + z \bullet 1$$

Looking at the above two equations, I recalled that the first term of the first equation is the square of the first term of the second equation, that is, $(x \bullet 10^2)^2 = x^2 \bullet 10^4$. I also noticed that $x \bullet 10^2$ is the largest component of the square root, assuming x is not equal to zero. Would $x^2 \bullet 10^4$ be the largest component of the square, 83,521? Suddenly I had an idea: maybe x is the largest number such that $x^2 \bullet 10^4$ is less than or equal to 83,521. If this was correct, what would x be? I tried some possibilities:

$$2^2 \bullet 10^4 = 40{,}000$$
$$3^2 \bullet 10^4 = 90{,}000$$

I could see that 2 was the largest possible x such that $x^2 \bullet 10^4$ remained less than 83,521. At this point, $x = 2$ was the best candidate.

Now how could I find y? Assuming for now that 2 was the correct value of x, I modified my image in the following way:

$$83{,}521 = 40{,}000 + (2x \bullet 10 + y)\, y \bullet 10^2 + (2x \bullet 10^2 + 2y \bullet 10 + z)\, z$$
$$43{,}521 = (2x \bullet 10 + y)\, y \bullet 10^2 + (2x \bullet 10^2 + 2y \bullet 10 + z)\, z$$

With this new data in sight, my previous idea still in mind, and my what-wonder driving me, I suddenly had another insight! My own *Aha!* I had made a connection, a leap, and with it the tension of inquiry was relieved and replaced with excitement. "I *think* I've discovered the answer!" Then spontaneously I began 'pulling together' the relevant data, the clues, and my possible what-answer to formulate my own clearly understood solution to the puzzle: I could find the remaining variables in the same way I found x, by finding the largest number such that each successive term in the equation was less than or equal to the total, and then subtracting that term from the total. Then, once x, y, and z were known, I would have $R = x \bullet 10^2 + y \bullet 10^1 + z \bullet 1$, the square root of 83,521.

My formulation of this procedure was a personal inner achievement, a new understanding which I could express in the words written above, or *alternatively*, as the following algorithm.

To find the square root, R, of 83,521:

- find largest x such that $x^2 \bullet 10^4 \leq 83{,}521$
- subtract $x^2 \bullet 10^4$ from 83,521 (= 43,521)
- find largest y such that $(2x \bullet 10 + y)\, y \bullet 10^2 \leq 43{,}521$
- subtract $(2x \bullet 10 + y)\, y \bullet 10^2$ from 43,521
- find largest z such that $(2x \bullet 10^2 + 2y \bullet 10 + z)\, z \leq$ remainder
- then $R = x \bullet 10^2 + y \bullet 10 + z \bullet 1$

One thing to notice about this expression of my formulation, this *formula*, is that it could be easily memorized. But would that memorization be anything like my understanding? As I am beginning to appreciate, memorizing the formula could in no way replace my activity of asking questions, getting insights, and formulating my own clearly understood solution.

2. Is-ing

At this point a very spontaneous shift occurred in my wonder. No longer was the solution remote; I had an idea, now I found myself asking: *Is it correct?* The shift was in the aim of my question. This was not a desire for understanding as my what-wonder had been; it was a desire for a correct affirmation of my idea. I was looking to make a judgment – to be able to assert 'yes, my formulation is correct.'

I found myself spontaneously looking back to the data, but now I was not groping in the dark as I had been in my what-ing. I knew what I was looking for, the looking was *intelligent*. I was testing my formulation, 'weighing it against the data' as I used it to solve for $y = 8$, and $z = 9$. Now I had R = 289. I asked 'does 289•289 equal 83,521?' I wrote out the multiplication, solving 289•289 = 83,521, and by the time I had finished I had grasped: *yes, my formulation allows me to find R such that R•R = 83,521.*

Now I had reached a second insight, my is-insight. *"It works! I've got it!"* This was a reflective event; it occurred only after having looked back to the data with my what-formulation in mind so that I could test it out and arrive at an answer to my is-question. So this involved three levels of consciousness: *sensing*, *what-ing*, and *is-ing*.

From this insight came another shift. Now I could ask myself “is my formulation correct?” and answer “yes, I am confident that I have discovered the correct procedure.” I had made a judgment of fact, and I felt a joy of knowing the answer, of coming to know the answer myself; being able to make a judgment of fact was my own personal inner achievement.

I now had a procedure for getting the square root of 83,521, but I realized that I was not finished yet. I knew that my procedure worked in this one case, but would it always work? Was it indeed a *rule*? I was asking another is-question: *Is this procedure for getting the square root of 83,521 a rule that will work for getting the square root of any square?* Again, I was seeking a correct affirmation of my idea. And again, in order to answer ‘yes’ or ‘no’ in response to my is-question I found myself spontaneously turning back to the data, not groping in the dark but looking intelligently. I knew what I was looking for, for I had grasped that my procedure is probably a rule if it works in a few other cases.

I immediately began testing my procedure with a new example. I picked another perfect square: 177,241. Could I use the algorithm from the end of Part 1 to find the square root of 177,241? I tried it out: first I needed to find the largest x such that $x^2 \bullet 10^4 \leq 177{,}241$. I found x equal to four. Next I subtracted $x^2 \bullet 10^4$ from 177,241, using $x = 4$:

$$177{,}241 - 4^2 \bullet 10^4 = 177{,}241 - 160{,}000 = 17{,}241$$

Then I needed to find the largest y such that $(2x \bullet 10 + y)\ y \bullet 10^2 \leq 17{,}241$. Substituting four for x I had: $(80 + y)\ y \bullet 10^2 \leq 17{,}241$. By trial and error I arrived at y equal to two. Next I subtracted $(2x \bullet 10 + y)\ y \bullet 10^2$ from 17,241, using $x = 4$, $y = 2$:

$$17{,}241 - (2 \bullet 4 \bullet 10 + 2)\ 2 \bullet 10^2 = 17{,}241 - 16{,}400 = 841$$

Finally, I needed to find the largest z such that $(2x \bullet 10^2 + 2y \bullet 10 + z)\ z \leq 841$. Substituting four for x and two for y I had $(840 + z)\ z \leq 841$. I could see that z was equal to one. I had arrived at the square root: 421. It worked!

What about larger numbers? Without working through an example I could see that my algorithm would not work for numbers whose square root was greater than or equal to one thousand. Because I arrived at the algorithm by starting with $R = x \bullet 10^2 + y \bullet 10 + z$, where x, y, and z were whole numbers from 0 to 9, the largest possible R that could be found using my algorithm was 999. However, it seemed likely that I could easily come up with a new algorithm to deal with larger numbers by following the same method as in Part 1, that is, beginning with the square root, e.g. $R = w \cdot 10^3 + x \cdot 10^2 + y \cdot 10 + z$, multiplying out and messing a bit with algebra.

By this point I very spontaneously reached another insight: *"This is it! I've got it!"* And this reflective grasp, my is-insight, propelled another shift in me. I now had sufficient evidence to judge: my procedure probably works as a rule to find the square root of any number, S, which is a perfect square and can be written in the form:

$$S = a \cdot 10^4 + b \cdot 10^3 + c \cdot 10^2 + d \cdot 10 + e$$

where *a*, *b*, *c*, *d*, and *e* are whole numbers from 0 to 9. In order to make a stronger judgment I would need to do a mathematical proof. However, that effort would go beyond the aim of this exercise in which my interest is primarily methodological.

3. Conclusion

With these observations I can establish that the events I noticed going on in me as I puzzled away are consistent with the *Dynamics of Knowing* identified by Bernard Lonergan, and presented in the diagram from "Appendix A" of *Phenomenology and Logic*.[2] These basic dynamics seemed to be a spontaneous and natural presence in me as I worked toward a solution. There was my wonder that I could express in the question: '*What is the rule?*' and through puzzling there occurred a sudden event-in-me, my what-insight; I had an idea that led to my formulation of a possible answer.

With my is-questions that followed I asked: *Is it correct?* My question was an expression of my wonder as a desire for a correct affirmation of my idea and led to what can be called a reflective insight. The insight was reflective because it involved looking back to the data with an intelligent what-formulation in mind and an is-questioning. When it seemed to me that I had sufficient evidence, I moved to a judgment: *yes, my formulation is probably correct*.

In my effort to approach this exercise with a dual focus, to attend to my own experiences of wonder as I solved the square root puzzle, I have begun to recognize my sensing of data, my what-ing, and my is-ing together as my very spontaneous and natural activities of *knowing*.

Meghan Allerton is a student in the Master of Environmental Science program at the University of Toronto at Scarborough. She holds an Honours B.Sc. in Ecology and Evolutionary Biology from the University of Toronto.

---

[2] Bernard Lonergan, *Phenomenology and Logic: The Boston College Lectures on Mathematical Logic and Existentialism*, vol. 18, *Collected Works of Bernard Lonergan*, ed. Philip J. McShane (Toronto: University of Toronto Press, 2001), 319–323.