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REFLECTIONS ON PROGRESS IN MATHEMATICS

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1. Introduction

The beginnings of mathematics go back to ancient times. Tens of centuries before Greek mathematics, special cases of the sum of squares relation were known in Babylon, China, and India. Standard history texts also discuss early arithmetic and "pre-algebra." Later, in the 3rd - 4th century BC, Euclid attempted something that was radically new, namely, a fully rigorous and comprehensive geometry and number theory. The earlier discoveries for certain right triangles were then raised to a single general formula which now is called the Pythagorean Theorem.

In those earlier times, results were obtained by eccentric individuals, often working in considerable mathematical isolation. Many centuries have passed, however, and (eccentricity of mathematical research aside) the social situation is now quite different. In the 20th century mathematics was "transformed from a cottage industry run by a few semi-amateurs into a world-wide industry run by an army of professionals."[1] So, in contrast to the early times of mathematical discovery (where mathematics was available to only a few), there has emerged a global ongoing complex range of mathematically related disciplines, publications, institutions, conferences, and meetings.

The vitality of mathematics, however, "is conditioned

[1] V. Arnold, M. Atiyah, P. Lax, B. Mazur, eds., *Mathematics: Frontiers and Perspectives* (Providence: Amer. Math. Soc. for the International Mathematical Union, 2000) [hereafter *M:FP*], viii.

upon the connection of its parts."[2] What, however, are the "parts" and "connections"? Is there, perhaps, some general pattern to this ongoing enterprise? In other words, is there some recognisable order to the mathematical project, not as in something to be imposed, but an order that can be verified in actual works and collaborations?

A main purpose of this paper is to offer an answer to this question in the affirmative. For there is accumulating evidence for the existence of an eight-fold periodic sequence of functionally related zones of enquiry H1, … , H8 – where for the rest of this paper these zones will be called *functional specialties*. In particular, each functional specialty would seem to have its own main objective and to involve its own differentiated type of enquiry.

The overall pattern of specialties is somewhat analogous to an 8-term periodic sequence of homology groups, as found, for example, in algebraic topology. In algebraic topology, however, the typical group sequence of interest is "exact," and so elements that pass through the sequence are quickly annihilated. The sequence of functional specialties for Mathematics is quite different. Specialty zones of enquiry do seem to form differentiated groups of operations, with their proper objectives. But the cyclic structure not only need not annihilate elements, but would seem, rather, to constitute a principle of growth and unity. Results of one specialty are materials for the next. And there would also seem to be vital cross-over relations between the various zones.

A detailed analysis of the periodic sequence of functional specialties in mathematics is not within the scope of the present article. The purpose of this article is to offer merely preliminary evidence, and so to raise the issue as a topic for possible further discussion. And since the question is intended to be generally empirical, suggesting the possible existence of the pattern is not suggesting that individuals or groups of investigators artificially confine their work to any one of the specialties. The initial result is, rather, that these specialties would seem to exist. Indeed, while some authors tend to favor one type of work over another, other authors would seem to

[2] D. Hilbert, quotation from *M:FP*, ix.

move from one zone to another, sometimes in a single paragraph.

As already mentioned, the contemporary situation in mathematics is, of course, both extensive and complex. There are textbooks in elementary mathematics, graduate mathematics, journals and periodicals emphasising certain areas of “pure” and/or “applied” mathematics, mathematics of physics, of chemistry, of biology, journals and textbooks on mathematics education, mathematics and technology, the history of mathematics, the philosophy of mathematics, mathematics institutions and organisations, and so on. And even within what is sometimes called “pure” mathematics, in addition to subject classification, there seem to be “two cultures,” “mathematicians who regard their central aim as being to solve problems, and those who are more concerned with building and understanding theories.”[3] Furthermore, the results of these numerous areas are not independent, but can have influence on each other, sometimes through explicit reference and sometimes through the (often implicit) point of view of an investigator.

A secondary purpose of the paper, therefore, will be to give some first indication of how adverting to the eight specialty zones mentioned may be helpful. The specialties do not occur in isolation, but are functionally related, and so, as mentioned above, reveal a functional unity to Mathematics. Identifying the specialties, therefore, could ground “a coherent ordering of … zones … that could help shift the statistics of … efficiency.”[4] (Note my indebtedness to McShane for introduction to the functional specialties. See the last paragraph of this paper for more details.)

Following up on Atiyah’s analogy that mathematics is “run by an army of professionals,” armies need as much as possible to be familiar with the terrain, to be aware of possible strategies (both tried and new), to be in control of supply lines, and generally to have efforts well-co-ordinated. For the “Mathematics Campaign,” existence of the specialties suggests

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[3] W.T. Gowers, *The Two Cultures of Mathematics* in *M:FP*, 65.

[4] Philip McShane, *A Brief History of Tongue* (Halifax: Axial Press, 1998), 97.

the possibility of finding a strategic division of labour.

Whatever one's individual talents and dispositions in mathematics, in as much as findings are shared, results can be drawn into the developmental dynamic of functional specialization. The long-term possibility would then be for the mathematics community to slowly reach toward improved collaborative control, in ways that would hopefully more efficiently exploit the natural potentialities of the total mathematical enterprise.

2. Past and Future

A first, possibly evident, distinction is between work that is oriented toward the past and work that is oriented toward the future. The mathematics expositor, for example, devotes most of his or her effort to understanding what has been already discovered. Some professional mathematicians, on the other hand, devote much of their effort to finding new solutions, and in many cases to discovering new problems. Again, there are the mathematical social structures of the past and present; and there are the mathematical (and interdisciplinary) structures that may develop (or decline) in the future.

The two orientations are related. For "mathematics has shown a consistent ability to renew itself by a synthesis of preceding work and an infusion of new ideas."[5] Indeed, what has been learned in the past becomes material for the future. And what is discovered in the future can shed new light on results of the past.

3. Encountering the Past

H_1 Research

Early discoveries were recorded on stone, clay, and papyrus. Groups of scholars formed, sufficiently like-minded to establish schools and libraries; in Europe's medieval times, there were some of the first universities. Undeniably, however, the stories of individuals and communities have not followed any straight course, and each has had its ups and downs. In some cases, libraries were buried under the debris of natural

[5] M. Atiyah, *Preface* to *M:FP*, ix.

disaster, or worse, were destroyed by war. Still, artifacts have survived, and in special cases documents themselves have been preserved, cherished perhaps by some collector.

Archaeology emerged and has become its own professional discipline. But, while archaeology includes the retrieval of artefacts, within the context of the total academic enterprise its very name expresses an ulterior motive. For there is the need to rescue the recorded "logos."

In other words, one of the purposes of archaeology includes providing data not only on how people of previous times lived, but also on what they said. There is, then, the question of meaning. But meanings vary in discipline and in type. So there is also library science, whereby documents, manuscripts, journals, books and other sources are collected, ordered, catalogued.

The first functional specialty H_1 therefore is characterised by its focus on data. This is meant broadly, and so includes all possible types of data, whether stone, clay, papyrus, paper, Braille, PC file, spoken word, and so on. In the *OED*, one of the suggested usages of the word "research" is: "systematic investigation and study of materials, sources, etc, in order to establish facts and reach conclusions." So, while the name "research" can be used in many ways, in the present context *Research* will be the name used for the first functional specialty of Mathematics. We can then further distinguish *Special Research* as the work of assembling data relevant to some particular question, such as, for example, Hilbert's view on mathematical development. *General Research* would be the work of archaeology, museums, libraries, etc.

H_2 Interpretation

Within each zone of enquiry, individuals of course both experience and understand. But it is the "large-scale" pattern of the zones of enquiry that is presently at issue, within Mathematics as a whole. So, where the large-scale work of the first functional specialty Research is to provide data, the proper work of the second functional specialty H_2 is interpretation. Research then is not aimless accumulation of random data, but is done "in order to establish facts and reach conclusions." Ideally, the work of Research is to compile and order data in ways that could help those working in the follow-up specialty *Interpretation* both understand and express what previous authors meant.

Note that the relation of Research to Interpretation evidently has an inverse relation. For where Research seeks and orders data for Interpretation, Interpretation provides some guidance to Research on what ordering of what data might be significant. (A more profound grounding of both specialties will come from "contemporary general categories." These emerge in the fifth functional specialty that, in fact, reaches out to all zones.[6]

Without a doubt, the problem of Interpretation is profound. In addition to treatises of mathematical results, there have been influential works on the nature of mathematical understanding, mathematical education, mathematical learning. But what is it to understand mathematics, let alone philosophical statements on mathematical understanding? If one reaches some tentative understanding of a first author, to what extent is it possible to faithfully express that understanding to some further audience?

These are fundamental questions. But the fact remains that Mathematics was an "(on)going concern" long before hermeneutics was discovered as a science. The work of the second specialty therefore does not properly include such fundamental questions. In no way is this intended to diminish the profound significance or necessity of hermeneutics. The present purpose, rather, is to describe and locate a particular type of work that happens to go on in Mathematics, a task that naturally follows on the results of Research. That is, authors read the works of authors and frequently publish reports of their findings to peers. So, keeping questions of efficacy aside for the moment, a type of work occurs that in this paper is being called Interpretation.

H_3 History

Mathematics has been in the making for some time. Millennia have passed since early counting techniques and pre-geometry. As cultures developed, it became possible (and

[6] See H_5 Foundations, below.

sometimes occurred) that discoveries of individuals either were shared with contemporaries, or were preserved for people of a later time. In any case, the discoveries of the individual can become the possession of a group. The group need not be a "village community" or even consist of scholars living all at the same time. Indeed, a mathematical group of scholars can consist merely of individuals who have a common base of questions, discoveries, and concerns. One oddity of "time," perhaps, is that individuals of an earlier time can share their results with those of a later time. Mathematics, then, is in an ongoing community enterprise.

There have been clusterings of interest, stages of development, and unfortunate periods of decline. There is the task, therefore, of determining what was going forward, or not. And this reveals the existence of a third functional specialty, which in this paper is being called *History*. Where the second functional specialty focuses on interpreting the results of individuals, the third functional specialty is for determining lines of development of, and within, the mathematical community, of identifying periods of progress and decline, of explaining transitions. History, therefore, seeks to know, as comprehensively as possible, what in fact happened.

H_4 Dialectic

Imagine two friends who are asked to review the activities of their mathematics department over the last decade. A main objective is for them to determine a (partial) history of the department.

Their individual findings may mesh together very nicely. A past department activity familiar to one colleague may be unknown to the other. Pooling their resources, there is the hope, then, of obtaining a fuller account than either of them could manage on their own.

It may also occur, however, that renditions of some events may differ considerably. They may each have different mathematical points of view. Even if they are from the same mathematical area, one colleague may be a senior expert in a particular mathematical discipline, the other perhaps a more junior faculty member. So their grasp, or even awareness, of certain issues and colloquia may differ significantly. In as much as some differences can be accounted for by their respective stages of development then (in principle at least) the differences could be reconciled. Some differences, however, may (once they are reduced to their roots) be found to be incompatible.

Besides differences in the historical accounts of the department, there may have been also differences operative within the department itself. A particular group may have favoured one area of mathematics, with a corresponding influence on graduate courses and department funding. Again, a chair of the department may have subscribed to a "school of pedagogy," affecting classroom policies and teaching practices. And so on.

In general, then, not only can different perspectives and viewpoints give rise to differences in written histories, but there may be differences in the lived history of the community itself. Atiyah refers to one of these community differences: He suggests "Arnold as the inheritor of the Poincare-Newton tradition, and Bourbaki as … the most famous disciple of Hilbert. Bourbaki tried to carry on the formal program of Hilbert of axiomatising and formalizing mathematics. … . Each point of view has its merits, but there is tension between them."[7]

One of Gödel's answers to the question of axiomatisation is his Incompleteness Theorem. Among other things, his theorem proves that no single set of axioms can be a basis for all mathematical theorems. In fact, one consequence of his theorem is the existence of multiplicities of unbounded sequences of higher viewpoints. While Gödel's result does establish the naivety of Hilbert's dream of reaching a "singly axiomatised" mathematics, it does not negate the importance of axiomatisation. In a positive sense, it provides some clue for the role of axiomatisation in the development of mathematics. For, besides "horizontal" development within an axiomatic context, there is the possibility of breakthroughs to new and higher contexts. In elementary mathematics, for example, we see arithmetic subsumed by elementary algebra; elementary

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[7] M. Atiyah, "Special Article – Mathematics in the 20th Century"*Bull. London Math. Soc.* 34 (2002), 1-15; 5.

algebra subsumed by group theory; Riemann Integration subsumed by Measure Theory; and so on.

The two traditions to which Atiyah referred have had some bearing on differences in mathematics education. In fact, there would seem to be a number of not only distinct, but in some cases fundamentally irreconcilable points of view. One approach seems to be somewhat in line with the Bourbaki school, focusing on “logical deduction” from axioms. In basic Calculus, for example, a large proportion of student texts begin with definitions (of, for example, “limit,” “continuity” and “derivative”). In the sense of Bourbaki, this approach of starting a text with definitions is logically rigorous. Another approach, less common perhaps, attempts primarily to foster questions and insights that then lead naturally and secondarily to solutions, definitions, techniques, and the emergence of further viewpoints (known to exist from Gödel’s Theorem).

In the Introduction to *A Concrete Approach to Abstract Algebra* W.W. Sawyer states:

> In planning … a course, a professor must make a choice. (The) aim may be to (have) every axiom stated, every conclusion drawn from flawless logic, the whole syllabus covered. That sounds excellent, but in practice the result is often that the class does not have the faintest idea of what is going on. … On the other hand, … students (may be lead to) collect material, work problems, observe regularities, frame hypotheses, discover and prove theorems for themselves. The work may not proceed so quickly … but the student knows what (they) are doing, … has had the experience of discovering mathematics, … no longer thinks of mathematics as static dogma learned by rote, …. (is) ready to explore further on (their) own.[8]

It is well known that the “logically rigorous” approach has not proven to be pedagogically successful. Likewise, the derived approach of focusing on mere symbolic technique has

[8] W.W. Sawyer, *A Concrete Approach to Abstract Algebra* (Toronto: W.H. Freeman and Co., 1959; San Francisco, Dover Pub, 1978), 1.

also been found to be ineffective. These are matters of high concern to the Mathematical Association of America. In fact, there is a rapidly growing number of professionals working toward developing new adequate pedagogical principles. However, the precise nature of the solution (whatever that solution may be) is not yet part of the general community.

Evidently, much as two colleagues in a department may, with regard to certain issues, have fundamentally different points of view, there can be differences in the mathematics community that are fundamentally incompatible, exerting forces on the community that yield quite different results. Some points of view would seem to foster mathematical and community development, while other points of view would seem to be less beneficial, and even with the best of intentions can prove harmful.

It follows that the first three functional specialties (Research, Interpretation and History) do not fully account for possible encounter with the past. For there can be fundamental differences in histories, histories both written by community members and living histories of the community itself. But no new historical account will answer the questions posed by those differences. Reminiscent perhaps of Gödel's Theorem, results from History can set problems that cannot be solved within History itself. What are needed, therefore, are studies that "are historical in an unusual sense, namely, in virtue of a thematic direction which opens up depth-problems quite unknown to ordinary history."[9] What is called for then is a further viewpoint, indeed, a further specialty. In this paper, this fourth specialty is called *Dialectics*.

The challenge of Dialectics is the challenge of a deeper engagement with past and present achievement, an engagement that will include the effort toward identifying conflicts emergent from History, Interpretation, and Research. Dialectics also will seek possible resolution of these conflicts. Work in this fourth specialty will seek to differentiate between

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[9] E. Husserl, "Essay on Geometry," Appendix to *The Crisis of European Sciences and Transcendental Phenomenology: An Introduction to Phenomenological Philosophy*, trans. David Carr (Evanston: Northwestern UP, 1970), 354.

perspectives and viewpoints that can (in principle at least) be reconciled, and which not. Are there results from History or Interpretation that are involved in “singularities” – that is, some kind of internal conflict or self-contradiction, and so are in some way inimical to mathematical development? Can those inconsistencies be reversed and so can results be preserved, at least in part? Are other results, while perhaps incomplete, otherwise compatible with sources of data and performance at all levels? Which are the points of view that are essentially sound and allow for, or even promote, development; and which require modification?

This direction of questioning will need to be allowed its full reach. So when results and perspectives have been found to be consistent with possible development, or when inconsistencies have been removed, there is the further possibility of exploring implications and possible prolongations of such positions. Based on differentiations, directions, potentials already determined, what are some of the consequent lines of development? In summary then, investigating what has been achieved already, the work of Dialectics involves identifying sound positions and remedying unsound positions.

Finally, note that the results of investigators in Dialectics will, of course, also not be immune to differences. In psychiatry there is the need for analysts to be as much as possible aware of their own biases and blocks. The situation in Dialectics is somewhat similar. In order for Dialectics to be efficacious, therefore, investigators will need to perform a similar analysis of their own and each others’ results. The work of Dialectics therefore calls for openness, detachment, discernment - like friends from a department trying (in some friendly way, with doses of humour perhaps) to get to and reveal the roots and implications of their differences.

4. Meeting the Future

H_5 Foundations

The fourth functional specialty seeks to determine the best (and the worst) of what has been. In turning toward the future we may ask, what is the best possible? So, if one were looking forward to the future of a department say, it would be useful to have some understanding of general needs and potentials; of types of work that might go well together; of types of meaning; and even some grasp of the full human potential.

This reveals, therefore, a new (and future oriented) zone of enquiry, which in this paper is called *Foundations*. In the literature, the name "foundations" has been used in several ways, so before going on, a distinction: There are books with titles such as "Foundations of Topology," which are "foundational" in the sense of usefully providing a more or less complete axiomatic treatment of a particular range of theorems. There is, however, another meaning to the word "foundations." Where one may seek to determine what is logically first in any particular axiomatic context, one may also seek to determine the very categories and principles of development which not only shape the expansion of results proper to a given context, but also drive toward breakthroughs to new and higher contexts. The fifth functional specialty is concerned with foundations in this second sense.

The results of past achievement (carried to the present by the first four functional specialties) constitute an invitation. The person working in the fifth specialty takes this invitation personally. There is the work of seeking out all possible categories of "best-possible growth." Part of the purpose of Foundations, therefore, is a type of development that would include "pushing orientations forward heuristically but concretely, toward possible and probable relative invariants." Note, also, that in as much as invariant groups of operations of functional specialization are verifiable, then they too would be embraced by foundational categorisation.

As mentioned above,[10] foundational development reaches out to and grounds work in all zones of enquiry. Whether or not one makes a study of it, one's basic orientation influences one's directions and efforts. In all specialties, therefore, categorial development from Foundations would foster an emerging control of meaning. Note further, that when involved in the work of one of the other functional specialties, one's foundational stance will also need to be "relatively stable." For

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[10] See n.6.

in that case one's orientation is not a mere enrichment in itself, but also is a functionally operative basis for one's involvement in the work of the other specialty. At the same time, since progress in Mathematics is ongoing and collaborative, it can be expected that new results and materials from any of the functional specialties could provide new data that would call for a return to Foundations - bringing refinement, further development, or even revision.

Each investigator, then, will have some basic orientation. And in some cases there can be memorable turning-points, implicit discovery of new basic directions. As expressed by F. Kirwan: "I have loved mathematics ever since my father showed me my first mathematical proof (from Euclid: 'the three angles inside a triangle are equal to two right angles')."[11] Foundations seeks to discover and commit to the full potential of all such breakthroughs.

H_6 Policies

The fifth functional specialty is part of the forward oriented phase of Mathematics. To some extent, however, it involves a necessary withdrawal. The fifth specialty does not, for example, provide directives on particular issues; neither does it yield new mathematical theorems. Rather, as already discussed, the fifth specialty provides a grasp of, and commitment, to orientations. Following on Foundations, then, there is the need to surrender to the norms and criteria of one's chosen orientation. In other words, there is the need to work with "the issues at hand." The sixth functional specialty is called *Policies*, and its function is to begin a direct engagement in that next forward part of the process.

As it happens, individuals may look to knowing and doing new mathematics. But individuals may also look to knowing and doing their part in the functioning of the mathematical community. What is revealed, therefore, is a bifurcation in the functional flow, a double focus in the future oriented phase of the mathematical enterprise. For, while there is the question of ongoing discovery of mathematical results, there is also the question of the ongoing structuring of the community, as a

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[11] F. Kirwan, *The Right Choice?* in *M:FP*, 117.

community.

Before working on particular new theorems, however, and before offering specific plans to a department regarding say, its organisation, it could be shrewd to first determine "basic guidelines" that pertain to the situation. In the structuring of a department, for example, there could be guidelines regarding "the mission of a mathematics department," "library needs and international communications," "teaching mathematics and the mathematical learning process," or even "humane principles for social groups," "group dynamics," etc, etc, etc. Regarding mathematical discovery itself, there could be counsel regarding worthwhile and promising new directions.

These "basic guidelines," or rather "policies," are of course not best deduced in isolation. The good consulting team will learn from past experience. So, reaching toward "true (and good) policies," the detailed results of Dialectics will necessarily come into play.

Again, we can do no better than go on from wherever we are; and it is the function of Dialectics to fully determine that part of the equation (in all of its implications). In response, Foundations develops and commits to "general field equations," as it were, on best possible fundamental directions. The sixth functional specialty Policies then makes a start from where we are; is grounded in and enriched by the general possibilities grasped by Foundations; and consequently works toward the development of guidelines, not only for worthwhile mathematical development, but also for community structuring that would promote worthwhile mathematical development. In short, Policies seeks to determine "reaching, relevant, pragmatic truths."[12]

Some mathematical examples might be useful. In geometry, Dialectics may conclude that Euclid's geometry had tremendous value, but that it suffered from certain deficiencies in method. It provided Mathematics with a first and extraordinary leap toward system and explanation. Euclid did not, however, clearly distinguish between description and

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[12] Philip McShane, *PastKeynes Pastmodern Economics – A Fresh Pragmatism* (Halifax: Axial Press, 2002), 62.

explanation. Consequently, there were certain logical difficulties.

The distinction between description and explanation, however, might be verifiable within the open context of one's foundational stance. One's developing orientation could then provide one with a basis for some grasp of geometric possibility in general. A resulting modest (but significant) mathematical "policy" might then be: *Euclidean Geometry is neither necessary nor self-evident*. Or, something in a more positive fashion: *Adequate axiomatizations of geometries (Euclidean and non-Euclidean) will have definitions that are free of description*.

The mingling of explanation and description has also directly affected the community. For instance, as revealed in Dialectics, there have been influences in mathematics education that have failed to distinguish mathematical understanding from a (mathematically empty and) merely descriptive understanding of symbolic technique. A possible corresponding community policy might be: *Good educational theories, plans, and institutions are verifiably in harmony with growth patterns of native intelligence; and in particular, they foster the emergence of mathematical (as opposed to merely descriptive) understanding*.

Certainly, policies may be understood from various points of view. (Functional specialization does not artificially confine the understanding of an investigator.) The functional role of "mathematical policies" from the sixth specialty, however, is part of the present question. But, following on Foundations, and prior to explanation, there is the possibility of description. The role of Policies, therefore, is well-defined.

In physics, seen light is, to a large degree, similarly described from age to age. Explanations of light, however, have improved with theoretic advance. In a somewhat similar way, by virtue of being descriptive, some policies in some contexts could also be relatively constant. Subsequent explanatory accounts of such constant policies would, though, be provisional, open to ongoing development and revision. This, however, would lead us into the work of the next specialty.

H_7 Systems-Planning

There is, therefore, further work to be done. For following on descriptive policies, we may ascend to an explanatory investigation, appreciation, and elaboration. It follows that there is a seventh functional specialty, which will be called *Systems-Planning*.

If there are directives and counsel on further geometry, what are examples of possible geometries, worked out in accord with best available geometry policies? Thus there is the ongoing development of new mathematical results, with explicit policies serving as helpful signposts.

Again, if there are policies on mathematics education, what is mathematical development? In particular, how is one to understand emergence of new viewpoints, worked out in a context that is fully explanatory?

H_8 Executive Reflection

It is one thing to have reached an explanatory understanding of possibilities. But choices need to be made. What in fact is to be done? This determines a selection-problem that defines the eighth and last functional specialty, *Executive Reflection*.

With regard to ongoing mathematical discoveries, there is the problem of expressing what has been discovered in the previous specialty. What one knows, one may also express. And in general, one expresses less than one knows. So expression requires a selection. What theorems will be published? What will be one's actual contributions? What results will be communicated to one's colleagues in the world community of mathematics scholars?

The world mathematical community, though, is structured. There are institutions, agencies, journals, conferences, all dynamically interlinked. There is, therefore, the actually functioning order of the community. This order, however, is an ongoing project, open to revision.

The object of Executive Reflection, therefore, continues the double-focus on mathematics and the mathematical community. For while selection of mathematical results for communication would contribute to the deposit of mathematical knowledge, this communication occurs within the context of the actually functioning, concretely ordered mathematical community.

Executive Reflection calls on the accumulated understanding and wisdom of the previous seven specialties. There is the need for selections that would contribute to the advance of both mathematics and the mathematical community. Executive Reflection, therefore, seeks the productive continuation of the mathematical collaborative enterprise.

Executive Reflection is the last of the functional specialties. For selections made in this eighth specialty will determine data that would be material for Research. And so the process cycles, and re-cycles.

Concluding Remarks

We are in a period of history where it may seem that all that one can do is at most keep up with the advances in some one or two specialized areas of expertise. One group of historians might focus on the origins of symmetry groups; one group of mathematicians might work primarily on certain problems in ring theory; one group of educators may enjoy certain types of field work in classrooms; and so on. And as is well known, this type of "subject specialization" has, for many, resulted in academic isolation. Adverting to and developing "functional specialization" promises to help break that isolation by allowing an investigator to be both in more control and to know better exactly how her work might contribute to the total collaborative mathematical enterprise.

Admittedly, already there have been certain notable achievements in interdisciplinary work. Interdisciplinary results in themselves, however, merely provide further instances of subject specialization, although hybrid in nature. Interdisciplinary results therefore provide additional rich material that needs to be included in a comprehensive study of progress.

In addition to the fact that functional specialization can be conceived as an intelligible and coherent model, there seem to be significant ways in which it is verifiable. Is one to attempt interpretation without having access to significant data? Can mathematical histories be written without knowing what individuals meant? The usefulness of collaboratively seeking sound positions cannot be denied. Perhaps it will be claimed to be a nice idea but not actually possible. Are we then to agree with that? If one is to deny the possibility of identifying categories, then by implication one has already determined the range of possible categorial development, for how else to discuss limitations to such development? So, unless one is to engage in an unfortunate type of self-contradiction, the alternative is to do one's best at foundational development and then to (at least provisionally) commit to the resulting norms and criteria. Before advancing to the development of new mathematics and new community plans, could there not be wisdom in seeking relevant pragmatic counsel? Once there are new discoveries and possible plans, there is always the need to select. And finally, all selection enters into the dynamic concrete structured community, and so produces new data.

Besides the differentiated work of each specialty, and its function relative to subsequent specialties, the total functional unity implies the existence of numerous cross-correspondences. For example, historical knowledge of what has already transpired could be relevant in the development of wise and good policies regarding what is to transpire. Again, consider the relations that will exist between Research and Interpretation.[13]

Functional specialization is, of course, not new. Features of the natural division of labor are implicitly alluded to in Husserl's paper quoted above. For a fuller presentation of the relevant quotations, see McShane's *Pastkeynes Pastmodern Economics*.[14] Also mentioned by McShane in the same text (p. 60) is the fact that "Arne Noess, the father of the Deep Ecology Movement, recognizing (the) disarray (in the movement), arrived at four collaborative layers that correspond roughly to the four forward-reaching tasks described above."[15] Furthermore, a main purpose of chapter 3 of *Pastkeynes* (again,

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[13] See H_2 Interpretation and footnote 6.

[14] *Pastkeynes*, 63-64

[15] *Pastkeynes*, 60; A. Noess, "Deep Ecology and Ultimate Premises," *The Ecologist*, 18, (1998), 131.

same text) is functional specialization in economics. So there is now ample evidence that the eight-fold division of tasks is relevant to the general academic enterprise. As mentioned in the Introduction, however, the present paper is intended only to introduce the possibility of functional specialization within Mathematics. A more comprehensive investigation of source evidence and other issues would certainly be needed.

In 1878, Felix Klein discovered "that a certain surface, whose equation (in complex projective coordinates) he gave very simply as $x^3y + y^3z + z^3x = 0$, has a number of remarkable properties, including an incredible 336-fold symmetry. He arrived at it as a quotient of the upper half-plane by a modular group … Since then, the same structure has come up in different guises in many areas of mathematics."[16] The name *The Eightfold Way* was given to a sculpture of Klein's group[17] because of the eightfold tessellation obtained "after the surface was folded over itself."[18] There is also "The Eightfold Way" in theoretical physics. Discovered by Murray Gell-Mann and Yuval Ne-eman, the "eight" here refers to the number of generating commutation operators of the symmetry group for strong nuclear interactions.[19]

As analogies, both of these Eightfold Ways seem relevant to the division of labor envisioned by functional specialization. Like the method of physics, functional specialization is an empirical process that yields ongoing cumulative results, especially in light of new data from the field. Like Klein's equation, the eight functional specialties would seem to be deceptively easy to describe, but potentially would admit numerous internal correlations of wide application. Moreover, the breakdown into two phases of inversely matched zones would seem to correspond to a mirror quotient group structure of a normative four-level ascent from data through to viewpoints.

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[16] Silvio Levy (Ed.), *The Eightfold Way – The Beauty of Klein's Quartic Curve* (Cambridge: Cambridge UP, 1999), ix.

[17] Ibid, Plate 1 following 142. Ferguson's sculpture is on exhibit at the MSRI, Berkeley, California.

[18] Ibid, ix.

[19] Murray Gell-Mann and Yuval Ne'eman, *The Eightfold Way* (New York: Benjamin., 1964).

Unlike the Eightfold Ways of Klein and Gell-Mann - Neemann just described, the eight-fold way of functional specialization reveals a beauty and structured unity to mathematical progress itself (progress that in the Systematics phase and seventh specialty generated Klein's particular mathematical group).

In *Recent Developments in Integrable Curve Dynamics*[20] Calini discusses the "self-induced dynamics," "self-energy" and "core acceleration" of certain "integrable vortex filaments." Functional specialization would reveal the self-energy of Mathematics, and that Progress in Mathematics is a self-induced dynamic core accelerating integrable eightfold community vortex.

In conclusion, I would note my indebtedness to Professor McShane, an indebtedness that will be evident to those familiar with his work on functional specialization in various areas: musicology,[21] literary studies,[22] linguistics,[23] and economics.[24] It seems to me that he has enlarged considerably the significance of Lonergan's discovery of the division of labor relevant to theology. He has, indeed, shown that functional specialization meets the emergent needs of all areas of inquiry, and that it grounds an academic ethics.[25]

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[20] Annalisa Calini, in *Geometric Approaches to Differential Equations*, eds., P.J. Vassiliou and I.G. Lisle, *Australian Mathematical Society Lecture Series*, 15 (Cambridge: Cambridge UP, 2000), 56-99.

[21] Philip McShane, *The Shaping of the Foundations – Being at Home in Transcendental Method* (Washington: UP of America, 1976), chapter 2.

[22] McShane, *Lonergan's Challenge to the University and the Economy* (Washington: UP of America, 1980).

[23] McShane, *A Brief History of Tongue* (Halifax: Axial Press, 1998), chapter 3.

[24] McShane, *Economics for Everyone* (Halifax: Axial Press, 1998; Edmonton: Commonwealth Publications, 1996), chapter 5.

[25] McShane, *Cantower 18*, <http://www.PhilipMcShane.ca/>, 2002.