Terrance J. Quinn: "The Calculus Campaign"
Journal of Macrodynamic Analysis
2 (2002): 8-36
http://www.mun.ca/jmda/vol2/calculus.pdf

THE CALCULUS CAMPAIGN

TERRANCE J. QUINN

I will discuss some of the difficulties that I have encountered in teaching Calculus. I will follow this, in Part I, with certain examples that my students have been finding helpful in reaching a preliminary notion of *derivative*. The focus in Part II is the genesis of the Fundamental Theorem of Calculus.

Introduction

Over the years I have become increasingly aware that there are problems with the standard textbook approach to teaching calculus. That there is a problem can show up in many ways, and in particular, in courses like Differential Equations, for which Calculus is a prerequisite. I have often taught third and fourth year courses in differential equations. The audience for these courses has varied. In some cases the course was intended for mathematics majors and concentrated on proofs and theoretical development; in other cases for physicists and engineers and concentrated on physical and technological applications; and in other cases for secondary certification students specializing in mathematics. The same problem, however, has consistently emerged in all of these student groups.

What I am speaking of, frankly, is a lack of basic understanding in Calculus. With that said, please know that the kind of *basic* understanding that I speak of is not the further more specialized understanding needed to generate proofs in Advanced Calculus courses. And it is not merely an inconvenience. Based on several years of teaching experience, it has become evident that this lack in understanding undermines the possibility of elementary competence, whether in merely computational applications or in theoretical development.

For instance, in one third-year differential equations class, I was teaching motion of a mass-projectile. Part of the problem was to integrate an equation. Together, we worked through to an answer. That there was a problem was revealed when I asked the students particular questions about the motion of the projectile. In friendly and candid discussion we discovered that while most were at least familiar with some of the differentiation and integration formulas, several did not know how to apply these formulas; and even some of the "A - Calculus students" admitted that they did not really know what these formulas meant. It is noteworthy that, in particular, while most of the students could reproduce the general formula $\frac{d}{dx}(x^n) = nx^{n-1}$, none of the students could give either reasons or examples regarding the derivatives of x^2 and x^3.

That there are inadequacies with the standard textbook approach is of course well known. As already described, one common situation is where the "Calculus Graduate" remembers some of the symbolism but is otherwise unable to solve particular problems. To address this issue in the United States, various studies have been done. Results of these studies have included the Harvard Calculus Reform, together with a number of follow-up textbooks intended to be in keeping with the precepts of the Reform. It is not my purpose here, however, to enter into a study of Calculus Reform as such. Note also that I will leave to a further paper any discussion of axiomatics, proof, or other possible generalities.

Let's look instead to the beginnings of the story. *The Calculus* was discovered in the 17^{th} century, by both Newton (1642 - 1727) in England and Leibniz (1646 - 1716) in Germany. Their discoveries led to solutions of what at that time were outstanding problems in mathematics and physics. In particular, the tremendous success of Newton's mechanics and theory of gravitation, formulated in terms of accelerations, is well known. But these initial discoveries were only the first stage of a long and fascinating campaign. In military terms, Newton and Leibniz established a beach-head. It was several decades, however, before a follow-up advancement occurred.

Like any discovery, in addition to answering questions, the results and techniques of the early Calculus also called forth new questions. In particular, there was the notion of *limit* that, while eminently useful, needed definition. This problem was solved by at least three mathematicians, namely, Cauchy (1789 - 1857), the Czech mathematician Bolzano (1781 - 1848) and the Portuguese mathematician Anastácio da Cunha (1744 - 1787), all of whom discovered essentially the same definitions of limit and convergence, with Anastácio da Cunha doing his work as early as 1782.[1]

Following on Cauchy's discoveries, Riemann (1826 - 1866) further advanced the front-lines of Calculus by discovering a definition of "area" and other integrals. When it exists, the *Riemann Integral* of a function is then a limit (as in Cauchy's work) of "Riemann sums" (over partitions of diminishing norm).

The developments of Cauchy and Riemann are, however, beyond the scope of the present article. The prior discoveries of Newton (1642 - 1727) and Leibniz (1646 - 1716) were made long before the work of Cauchy and then Riemann. And with regard to teaching Calculus, I have found that students (mathematics majors and non-majors alike) can enjoy being guided along a similar path. In particular, I have found that introducing definitions of *limit* and *limit of Riemann sum* too soon can leave students more than a little puzzled and wondering, for example, why The Fundamental Theorem is called *Fundamental*.

The advancement and envelopment that comes with definition can be an ultimate objective, especially for the mathematics major. On the other hand, students have often expressed to me their delight with the basic understanding reached by starting with simple examples of advancing areas and front-lines. So the primary purpose of this article is not a

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[1] The works of Bolzano and Anastácio da Cunha "appeared in the far corners of Europe and were not appreciated, or even read, in the mathematical centers of France and Germany. Thus it was out of Cauchy's work that today's notions developed." Victor J. Katz, *A History of Mathematics: An Introduction* (New York: Addison-Wesley: 1998), Ch. 16.

development of the subject based on the explicit formulation of the definitions of Cauchy, Riemann, et al., and as would be called for in a course in Advanced Calculus. My purpose, rather, is to encourage the implicit and definite beginnings of a first understanding that can be had by a beginner, whether the beginner is mainly interested in applications or is hoping perhaps to do later work with axioms and proofs. I have found that by following this approach students typically reach a command of the general formulas for themselves; and for the mathematics major, the need for definition can become poignant.

Part I: Ratios of Change – The *Derivative*

The main objective of Part I is to use "increasingly accurate" ratios in order to determine a "rate of change". The examples are of certain domestic situations that my students have found engaging.

1. The Sneaky Farmer

Suppose that farmer Sam say, has a square field, 100 yards by 100 yards. Two adjacent sides of the field are bordered by straight roads, and the other two sides are bordered by the large fallow pastures owned by farmer Frank.

Sam is not happy with the situation, for he would like to own more property. But the roads, together with Frank's pastures, have Sam's square property hemmed in. What to do?

Sam has his clever moments; and it is not past him to be somewhat sneaky. He is even capable of being a little dishonest, if the need arises. So Sam devises a plan. He decides to increase the size of his property "little by little", hoping that by doing this slowly enough, his neighbor Frank won't notice. Sam is patient. He plans, in fact, to work his scheme over a period of one year.

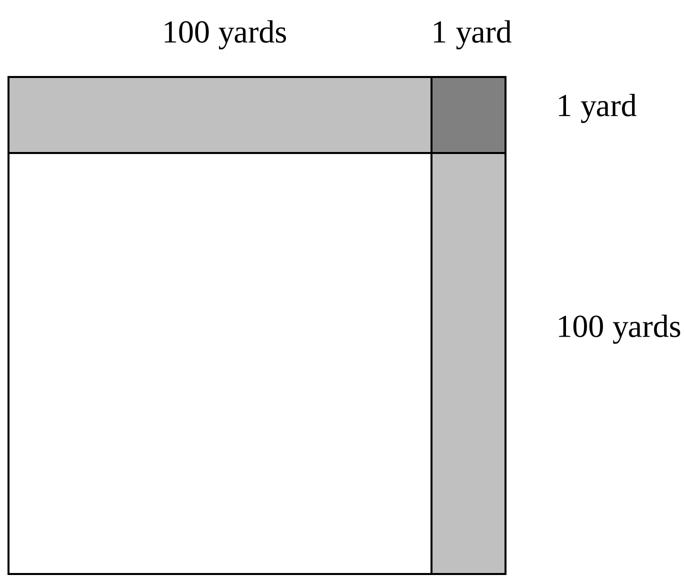
Here is what Sam plans to do under cover of darkness. The first night he will sneak out to the property. He will mark one extra yard along the road-sides of his property. There are property marker posts both where the properties meet the road and at the far corner away from the roads - so three marker posts in all. He plans to dig three new holes and then move the marker posts to the new holes.

Sam will then wait one month. After one month, the original post-holes will be well covered with growth, and he will then go on to repeat this process for twelve months. At the end of twelve months Sam hopes to take what he feels will be a well-earned rest.

The question now is: At what rate will Sam be increasing his property?

We can start by looking at a diagram for the layout of property.

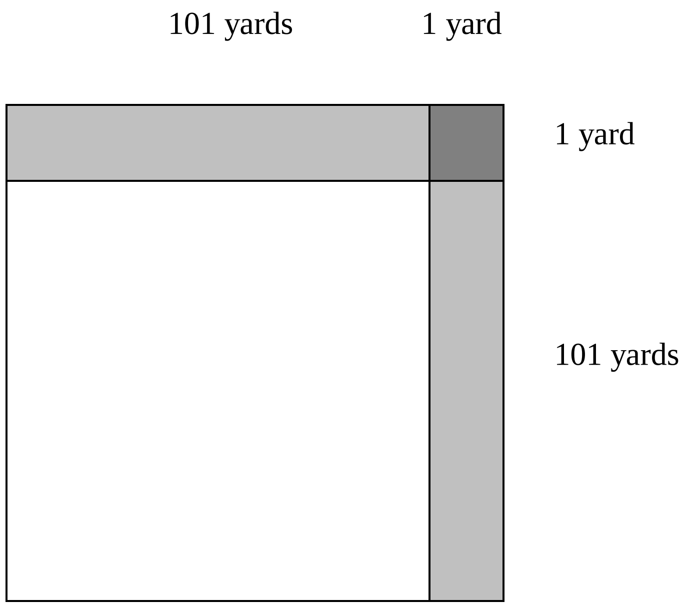
At the beginning of the first month, Sam's property is 100 yards by 100 yards. Because of Sam's "extension program", at the end of the first month the marked field will be 101 yards by 101 yards.



Looking to a diagram, the added property comes from the rectangles along each edge together with the square corner. Following convention, this new property can be calculated to be (100x1) + (100x1) + (1x1) = 2(100) + 1 square yards.

Let's follow Sam in his scheme. The second month, his marked property will change from 101 yards by 101 yards to 102 yards by 102 yards.

Again, by looking to a diagram, we can calculate the added property to be (101x1) + (101x1) + (1x1) = 2(101) + 1 square yards.



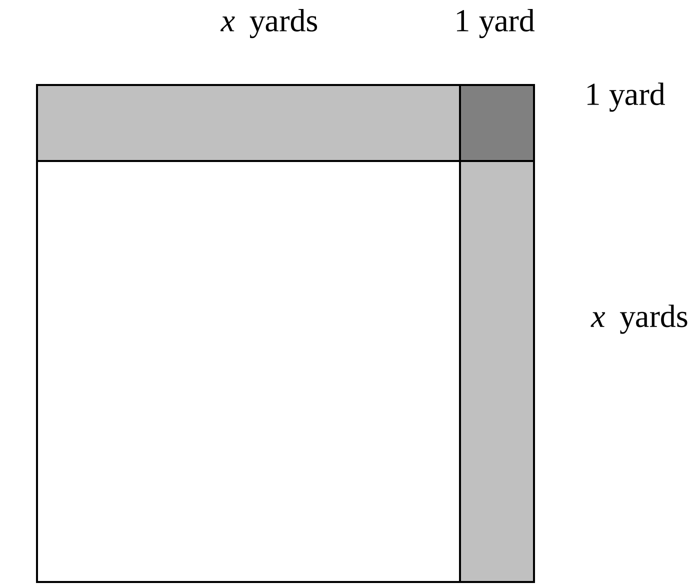
Do you have a pattern yet? Let's do another month. I leave the diagram as an exercise.

At the beginning of the third month, the field will be 102 yards by 102 yards. Moving the marker posts, the field will then become 103 yards by 103 yards.

The added property is (102x1) + (102x1) + (1x1) = 2(102) + 1 square yards.

So at each stage, except for the 1 square yard at the corner, the monthly contribution to new property will be 2 times the straight length of the square border of Sam's property with Frank's property.

Can we bring more precision to this? Using algebra, let's suppose that, at the beginning of a month, the length of the field marked out by Sam is x yards by x yards. He then changes the markers to give $x+1$ yards by $x+1$ yards.



So, using the conventional formulas for area, the added property is $(x\times 1)+(x\times 1)+(1\times 1)=2x+1$ square yards. In other words, except for the square yard at the corner, the monthly rate at which Sam gets new property is 2 times the boundary of the square field, that is, the monthly rate is $2x$.

2. The Sneaky Apprentice

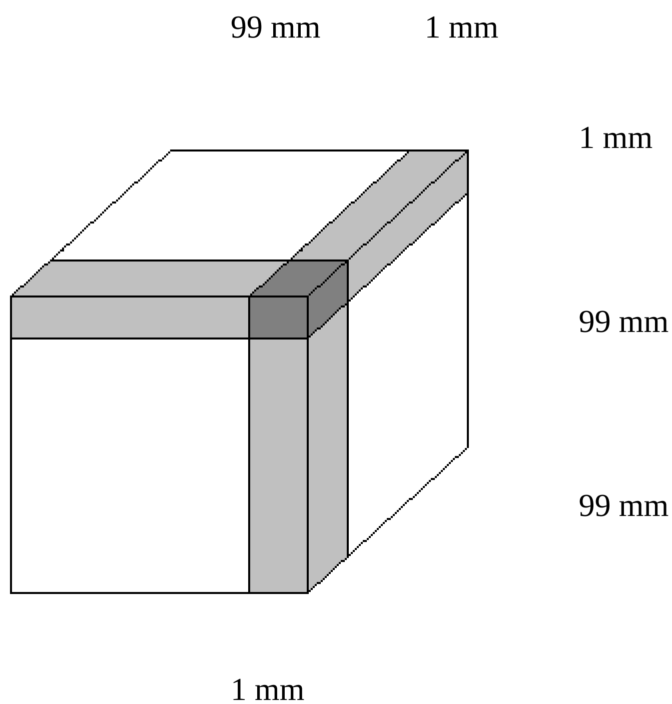
In this example, suppose that an apprentice Al works in a metal workshop. Al is very keen on working with metal. In addition to what he does at his master's workshop, Al also has several projects of his own that he works on at home in his free time. One project that is especially dear to him requires quantities of lead. As luck would have it, there is a cube of lead at the workshop. Unfortunately, somewhat like his country cousin Sam, Al is not always honest. So Al too decides on a somewhat devious plan.

The cube of lead at the workshop is 100 mm by 100 mm by 100 mm (mm = millimeters). Al hopes that his master Mack won't notice small changes to the cube. In fact, there is a device at the workshop that might be handy for this. The block of lead can be placed in a steel corner. There are hot blades that can then be used to shave lead off of each of the three exposed sides.

On each occasion, Al plans to remove 1 mm from all three sides. That way the cubic shape will be preserved and the changes will hopefully go unnoticed. Like his cousin Sam, he plans to do this only once a month.

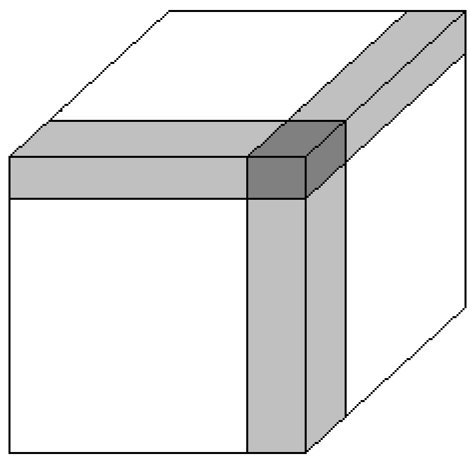
Again, the question is, what is the monthly rate at which Al will get lead for his home projects?

At the beginning of the first month the cube of lead is 100 mm by 100 mm by 100 mm. After Al shaves the cube, the remaining lead will measure 99 mm by 99 mm by 99 mm.



Now, the convention for calculating the quantity of a solid is to multiply the measured lengths of perpendicular sides. (This is usually called *volume*.) Looking to the diagram, the lead that Al will take will consist of three cut portions from each square surface, three narrow rectangular edges, and one cubic mm at the common corner. So, in total, Al takes away 3(99x99x1) + 3(99x1x1) + 1(1x1x1) = 29,403 + 297 + 1 cubic millimeters of lead.

The next month Al will repeat the process.

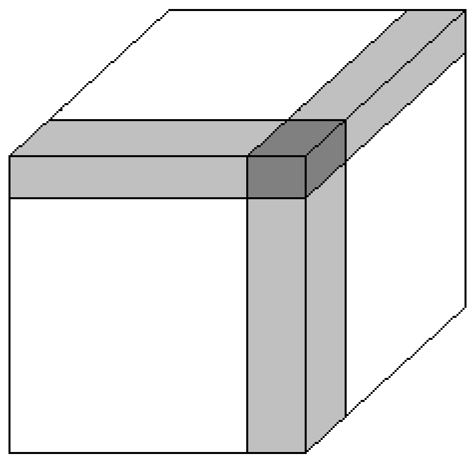


This time the amount of lead that Al takes way is 3(98x98x1) + 3(98x1x1) + 1(1x1x1) = 28,812 + 84 + 1 cubic millimeters of lead.

Do you see a pattern?

The main contribution to the lead that Al is taking away comes from the three exposed square surfaces of the cube. Again, using algebra, we can be more precise. Suppose that the cube of lead measures x mm by x mm by x mm.

If Al then removes 1 millimeter from each of the three exposed sides; he will take home $3[(x-1) \times (x-1) \times 1] + 3[(x-1) \times 1 \times 1] + 1[1 \times 1 \times 1] = 3x^2 - 3x + 1$ cubic millimetres. In other words, taking lead from the three exposed x by x square surfaces (each of which is x^2 cubic millimeters), the main contribution to the monthly rate at which Al removes the lead would be $3x^2$.



3. Refinement of the Rates

(a) The Farm Field

Suppose that Sam the Sneaky Farmer reconsiders his original plan. His desire to have more property remains, but he suspects that, after all, surely his neighbor Frank would notice if the field markers were changed by a full yard each month. So, taking a more cautious approach, Sam decides to increase his field, not by 1 yard each month as originally planned, but by 1/3 of a yard (1 foot) each month.

Let's now ask the same question as before: What is the monthly rate at which Sam will be increasing his claimed property?

As before, at the beginning of the first month, the dimensions of the field are 100 yards by 100 yards. Once the length of the field along the road is changed by 1/3 yard, the new field will then be (100 + 1/3) yards by (100 + 1/3) yards.

As before, we can look to the diagram (exercise). The added property comes from the two outer rectangles along the field lengths together with the square corner furthest from the road junction. Calculating, Sam would obtain an additional (100x1/3) + (100x1/3) + (1/3x1/3) = 2(100 x 1/3) + 1/9 square yards.

Sam now wants to know how efficient this is, in other words, he wants to know how much is he getting for his effort. One way to answer this question is to calculate the ratio [added property] to [change in length], that is, the change in property per change in length. Using the above sum, we get [2(100 x 1/3) + 1/9] divided by [1/3], which reduces to 2(100) + 1/3.

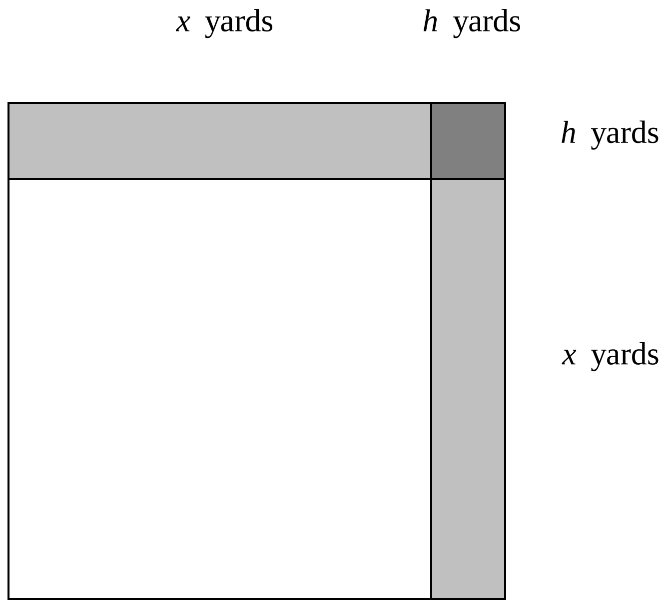
In exactly the same way, if we look at what happens in the second month, we get a ratio of [2(100 + 1/3) x1/3) + 1/9] divided by [1/3] which reduces to 2(100 + 1/3) + 1/3.

As with Sam's original plan, again let's see what algebra can reveal. If the original length of each side is x, and this is increased by 1/3, then the added property is $(x\times 1/3)+(x\times 1/3)+(1/3\times 1/3)=(2x\times 1/3)+(1/3\times 1/3)$. So the ratio [additional ground cover] to [change in length] would be $2x+1/3$ square yards of property per yard changed in property length. This is much the same as the result from Sam's original plan. But this time, the approximate rate is even closer to being exactly 2x.

Comparing the two calculations, the smaller change in length corresponds to a ratio of change that is closer to being $2x$ (2 times the starting length x). To push this further, let's give the change in length a name, h say. (In Sam's first plan, $h=1$; in the revised plan, $h=1/3$.) But, now using h in place of any particular number, perhaps we will be able to detect a general pattern of change.

If the original length is x, and this length is changed by h, then the new length would be $x+h$. Just as before, the added property will come from the rectangles along the edge of the new square field together with the corner that is h yards by h yards.

The ratio [additional ground cover] to [change in length] would then be $2x+h$. So, as anticipated from the numerical examples, the smaller h is, the closer the ratio of change $2x+h$ is to being $2x$.



(b) The Cube of Lead

Can we now get some similar type of result for the cube of lead? For Al too decides to be more careful. Instead of 1 mm per month, he decides to take ½ mm of length each month.

Remark: At this point (having completed the square field example), I have found it can be a good exercise for the student to do some numerical calculations for the cube by themselves. Most of the time, the student is already onto the game. They compare the two cases and find that, for a starting length of x, the ratio [change in quantity of lead] to [change in length] is closer to $3x^2$ for the smaller change in length ½ mm. Again, a key point here is that the x^2 term is the quantity of lead from each face of the cube. The factor of 3 comes from there being 3 faces where the change takes place.[2] Following this exercise, I usually jump to the ratio of change of lead for an arbitrary (non-zero) change of length h. That is, the change in volume

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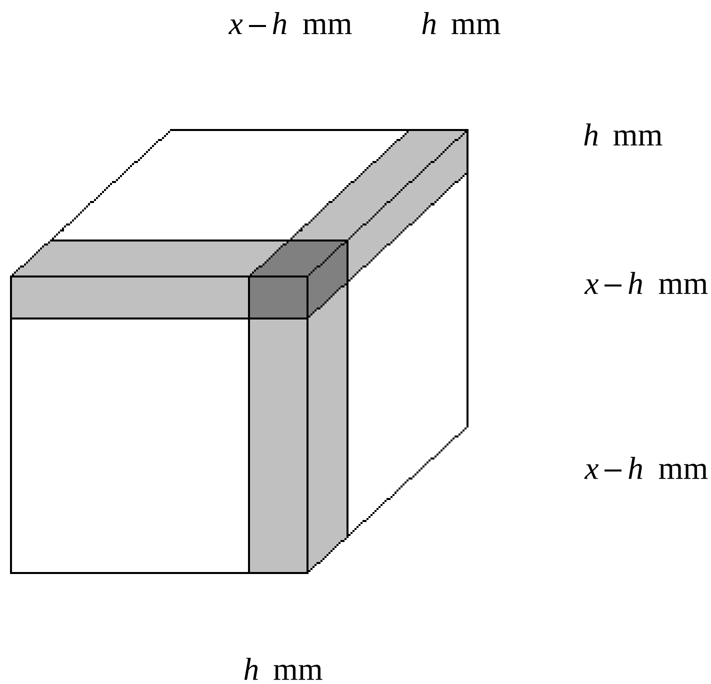
[2] This key point of course re-appears in the multi-variable calculus as Green's Theorem, The Divergence Theorem, and the other "Stokes" Theorems.

of lead given by $[x^3 - (x-h)^3]$ compared to the change in length given by $x-(x-h)=h$.

Suppose then that the cube that Al starts with is x mm by x mm by x mm, and that he shaves off h mm.



As can be seen from the diagram, the lead that Al obtains in this way comes from 3 cut faces, 3 narrow edges, and the corner piece. This adds to $3(x-h)^2h + 3(x-h)h^2 + h^3$ cubic millimeters of lead.

From the diagram, the main contribution to the lead that Al takes away comes from the three cut faces. Again, using algebra, we can be more precise:

The ratio of change [change in quantity of lead] to [change in length of the cube] is the ratio of $[x^3 - (x-h)^3]$ to $x-(x-h)=h$. Doing the calculation, this is $[3(x–h)^2h + 3(x–h)h^2 + h^3]/h = 3(x–h)^2 + 3(x–h) + h^2 = 3x^2 - 3xh + h^2$.

Keep in mind that x is a fixed number – the number x is whatever the starting length is for the sides of the cube; and the number h is whatever is removed from x. So the calculation reveals both that the ratio depends on how much is actually removed, and that the smaller h is, the closer the ratio of change is to $3x^2$.

Summary for Squares and Cubes

The Square

Suppose that a square has dimensions x by x. Then for small changes in length of the sides, the ratio [change in area] to [change in length] is approximately $2x$. That is, the ratio is approximately 2 times the length. The smaller the change in length, the closer this ratio is to being exactly $2x$.

The Cube

Suppose that a cube has dimensions x by x by x. Then for small changes in length of the sides, the ratio [change in volume] to [change in length] is approximately $3x^2$. That is, the ratio is approximately 3 times the surface area of each face. The smaller the change in length, the closer this ratio is to being exactly $3x^2$.

4. Higher Powers

Suppose that Ralph owns a property out of town where he makes a vegetable-based liquid fertilizer for gardens. Ralph sells this fluid and transports it in metal cubes. The cubic containers are made by a machine that, within limits, can be adjusted to any length from 1 foot to 5 feet. Since the larger cubes contain more liquid, they are heavier and require further reinforcement. So Ralph sells his fertilizer at a rate that depends on the size of the cube in which he delivers the cube. If a cube is 2 feet by 2 feet by 2 feet, then Ralph charges 2 dollars per cubic foot. If a cube is 2.5 feet by 2.5 feet by 2.5 feet, then he charges 2.5 dollars per cubic foot. And so on. So, within the limits of construction, for a cube that is x feet by x feet by x feet, Ralph sells his fertilizer at x dollars per cubic foot. That means that Ralph's revenue on a cube that is x feet by x feet by x feet is (the number of cubic feet of fertilizer) x

(x dollars per cubic foot) $x^3 \times x = x^4$ dollars.

Evidently, the revenue increases with the length of the cube. Ralph would like to know more. He would like to know at what rate his revenue increases with an increase in the length of the cubes.

Just as with Sam the farmer and Al the apprentice, we can compare the changes in quantities - in this case the ratio in question is [change in the revenue] to [change in length]. This will give an approximation to the rate at which the revenue x^4 increases per change in linear foot x. Motivated by our success with algebra in the previous examples, we can calculate this ratio explicitly. If x is a given length of cube, and the length is increased by h, then the change in revenue is $[(x+h)^4 - x^4]$. So the ratio of change is $[(x+h)^4 - x^4]/h$ = $[4x^3h + 6x^2h^2 + 4xh^3 + h^4]/h = 4x^3 + 6x^2h + 4xh^2 + h^3$.

As before, remember that x is a fixed number. So, from the algebra, the ratio depends on the change h; and the smaller h is the closer the ratio is to $4x^3$. In other words, for a relatively small h, Ralph would be increasing his revenue at a rate that is approximately $4x^3$ dollars per linear foot.

From here, students often jump to the generalization for higher powers of x. What we have so far is that for quantities of the form x^2, x^3, and x^4, a small change h in x produces rates of change that, respectively, are approximately $2x$, $3x^2$ and $4x^3$. Students will conjecture that for quantities of the form x^5, x^6, x^7, ... , a small change h in x will produce rates of change that, respectively, are approximately $5x^4$, $6x^5$, $7x^6$, ... Note that quantities involving higher powers can be illustrated using investment examples that involve compound interest.

Now it is one thing to make a conjecture (based on patterns in symbolism). Can we do more? Since algebra has been useful so far, can we do for the higher powers of x something like what we did for the first few powers?

Let's look again at the quartic x^4. The ratio of change is $[(x+h)^4 - x^4]/h$. In the above calculation, I calculated the numerator explicitly. Let's do this again, but this time let's not focus so much on getting the explicit result, but on determining the role played by h in the ratio.

The numerator is the change in the quartic $[(x+h)^4 - x^4] = (x+h)(x+h)(x+h)(x+h) - x^4$. The product of parentheses expands to gives a sum of products. Each product in the sum consists of some x's and some h's - but always four factors in total. The first term will be x^4, and this will cancel with the $-x^4$. Tracing the multiplication through the parentheses, there will be four ways to get terms of the form x^3h, and the rest of the terms will be of the form x^2h^2, xh^3 and h^4. But the ratio of change is obtained by dividing $[(x+h)^4 - x^4]$ by h. So the ratio of change will be of the form $4x^3$ + (a sum products of terms - each of which has at least one power of h). Again, remember that x is fixed. It follows that the smaller h is, the closer the ratio will be to $4x^3$. In other words, for small changes h in x, the rate of change of the quantity x^4 is approximately $4x^3$. Of course, we already have this result. But do you see perhaps how this approach can be applied to the higher powers?[3]

Let's test this approach on a power of x that is beyond easy explicit calculation. Suppose then that a quantity is of the form x^{10} say. Then a ratio of change is $[(x+h)^{10} - x^{10}]/h$. Writing this out as above, the numerator is $[(x+h)(x+h).....(x+h)] - x^{10}$, where the parentheses $(x+h)$ are repeated 10 times. Again, tracing through the multiplication, there will be one term of the form x^{10}, which cancels with the $-x^{10}$. There will be 10 ways of getting x^9h; and the rest of the numerator will be a sum of products, each of which has at least 2 powers of h. Calculating the ratio cancels one h in each product. The ratio $[(x+h)^{10} - x^{10}]/h$ is then of the form $10x^9$ + (a sum of products, each of which has at least one h). So, for small changes h, the rate of change of the quantity x^{10} will be approximately $10x^9$.

[3] Note that this approach does not require using the general formulation of the binomial theorem.

Student Exercises:

1. Using the above approach, explain why, for small values of h, the approximate value of the ratio $[(x+h)^{20}-x^{20}]/h$ is $20x^{19}$.

2. Suppose that n is a non-negative integer. Using the above approach, explain why, for small values of h, the approximate value of the ratio $[(x+h)^n-x^n]/h$ is nx^{n-1}.

5. A Common Denominator

We have been studying rates of change in quantities that are given by powers of x. In each case, we get a notion of a distinguished quantity. For instance, in the case of Sam the Sneaky Farmer, if his field is originally 100 yards by 100 yards, and if he were to increase the length of his field by h, then the ratio [Change in Property]/[Change in Length] = $2(100)+h$. As already discussed, the smaller the change in length h, the closer this ratio is to being exactly 2(100). This distinguished quantity 2(100) need not be an actual ratio of change; but actual ratios can be made close to this quantity, by choosing h to be small. Historically, it is this approximation to 2(100) that gave rise to the name *limit*. The distinguished quantity 2(100) is more precisely called the *limit* of $[(100+h)^2-100^2]/h$, as h goes to zero (that is, as h gets small). Our general result for the square field was that, for small h, the ratio $[(100+h)^2-100^2]/h=2x+h$ is close to the distinguished value $2x$. So the value of the *limit* depends on x, the starting value for the length.

This can all be a little tricky to write down. It is generally accepted that both Newton (1642-1727) in England and his contemporary Leibniz (1646-1716) in Germany independently discovered these *limits*. Where Newton used Calculus to establish a new physics, it is the notation of Leibniz that better represents the quantities involved and suggests further mathematical results (such as the chain rule and the product rule). Following Leibniz then, we obtain the following formulas: $\frac{d}{dx}(x^2)=2x$, $\frac{d}{dx}(x^3)=3x^2$, $\frac{d}{dx}(x^4)=4x^3$, and so on. The symbols $\frac{d}{dx}$ express the following: (a) we consider ratios of differences, hence the symbols "d" for "difference"; and (b) the answer need not be any particular ratio, but is a distinguished quantity, the quantity to which the particular ratios are close for small changes h in x.

PART II: A Lucky Advance – *The Fundamental Theorem* of Calculus

For Part II, the topic is the particular instance of rate of change determined by an advancing "area". The examples that I use are drawn from World War II and the liberation of Nazi-occupied Europe. My students have been enjoying the stories of the advancing front-lines and have been using them to make their own first breakthroughs toward The Fundamental Theorem of Calculus.

1. Normandy and Beyond

D-Day was June 6, 1944 -- Allied Forces landed on the beaches of Normandy and began Operation Overlord, the invasion of Nazi-occupied Europe. During late July and early August, the "Third Army spearheaded Operation Cobra, the great breakout from the Normandy bridgehead. In a matter of days, what had been a troubling and potentially deadly stalemate, turned into one of the most dramatic Allied victories of World War II. The German army in Normandy was shattered, and its survivors were forced to retreat in disarray, mostly on foot, behind the River Seine"[4]. Paris was soon liberated from Nazi occupation and army groups consisting of British, American and Canadian forces swept northward into Belgium and eastern France. At the same time, as part of the U.S. Twelfth Army Group under General Bradley, Patton's U.S. Third Army moved across southern Normandy and then eastward. Allied army groups were to converge later along the Rhine.

[4] John Nelson Rickard, *Patton at Bay: The Lorraine Campaign, September to December 1944* (Westport CT: Praeger Publishing: 1999), xi.

Famous in military history are *Lucky*'s[5] advances across Europe, despite extreme enemy opposition and difficult terrain. The Third Army, however, was dogged in its purpose. Overcoming numerous difficulties, in only 231 days Patton lead his Incredible Third[6] to victory over occupying Nazi Forces, liberation of the terrible death camps, and the liberation of European territories ranging from Normandy to Germany[7].

2. Advancement of a Front-Line

Gains in territory depended on circumstances in the field, and so varied from one battle to another. One question then is the following: Is there some way to determine the "rate" at which territory is obtained by an advancing army?

Let's look at the progress of Lucky's U.S. Fourth Armored Division in the Eifel campaign. In a startling advance, the Fourth Armored established the Trier-Koblenz corridor, breaking Nazi front-lines from Trier to the Rhine (Koblenz) in just three days.

For purposes of illustration, let's suppose that the width of the Fourth's advancing front-line was approximately 4 miles.

If this "4 mile" front-line was advanced 1 mile, then the territory gained would have been 4 miles in width and 1 mile in depth, that is, 4 square miles. Advancing the front-line a second mile, another 4 square miles would have been obtained. So, for an advance of 2 miles, the territory gained would be 4x2 square miles; and so on. In other words, one way that we can determine the "rate" at which territory is gained is by using the width of the front-line. In this example, where the front-line is 4 miles in width, the territory gained is 8 square miles per mile that the front-line is advanced.

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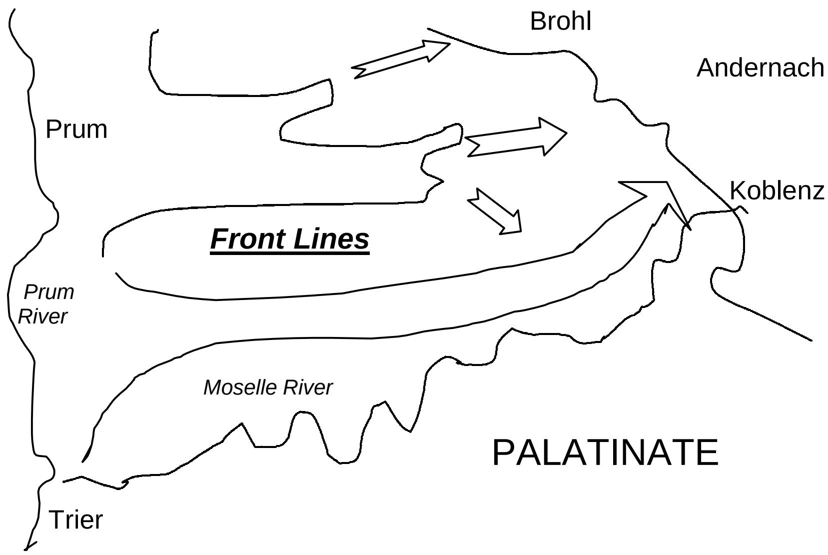
[5] Military groups were commonly given pseudonyms. "Lucky" was the name Patton gave to the Third Army.

[6] Gen. Paul D. Harkins, with Eds. of Army Times Pub. Co., *When the Third Cracked Europe - The Story of Patton's Incredible Army* (Harrisburg PA: Stackpole Books, 1969).

[7] Robert Allen, *Lucky Forward – The History of Patton's Third U.S. Army* (New York: The Vanguard Press, 1947), 395.

THIRD ARMY'S BREAKTHROUGH TO THE RHINE IN THE EIFEL



Scale: 0 --- 5 --- 10 MILES

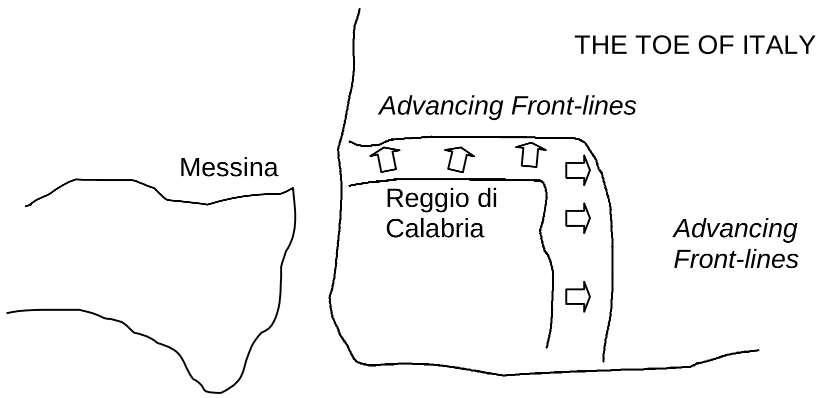
Of course, the actual width of front-lines was not usually constant, but would change through the course of battle. But, if the width changed, that change would have taken place in stages. So, even if the front-line width was not constant at 4 miles, for as long as it was approximately 4 miles across, the gain in territory would have occurred at a rate of approximately 4 square miles per mile advanced. As the front-line width was expanded to 5 miles, then for as long as that front-line width was approximately 5 miles, the rate at which territory was gained would have been 5 square miles per mile advanced. And so on. So, one answer to our question is that if territory is advanced along a straight front-line, then the rate at which territory would be gained would be given by the width of the front-line.

3. Expansion of Front-Lines

You may be thinking that front-lines not only change in length, but typically are not straight. So, for our next example, let's return to an earlier part of the war. After being taken by the combined forces of U.S. Seventh Army under Patton and

British Eighth Army under Montgomery, Sicily was the base for a jump-off[8] to mainland Italy[9]. On September 3rd, 1943, "two divisions of Montgomery's Eighth Army crossed from Messina to Reggio di Calabria and advanced up the Italian toe against slight resistance."[10]

MESSINA TO REGGIO DI CALABRIA



THE MEDITERRANEAN SEA

Again, our question is the rate at which territory is obtained by an advancing army. So let's imagine the advance of the Eighth Army as it pushed its front-lines across the toe of Italy. The territory was bound on the west by the shore-line of the Straight of Messina and on the south by the Mediterranean Sea. So imagine the expansion of Allied territory occurring in two directions at once, both north and east.

The region is almost square and so, again for purposes of

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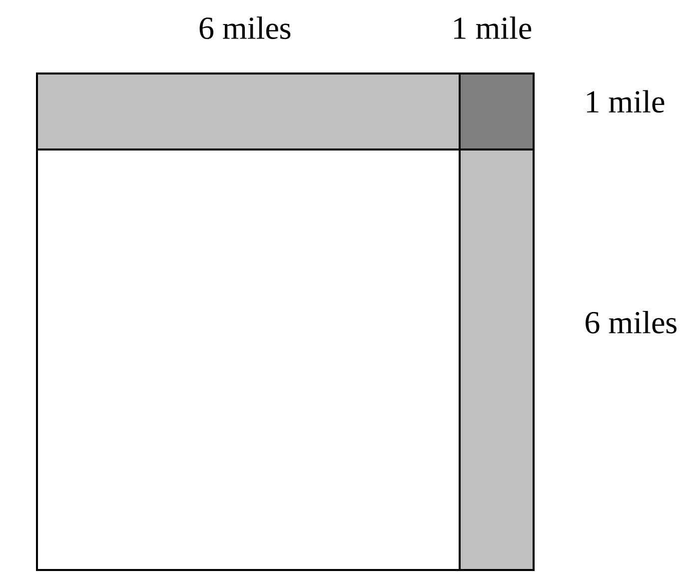
[8] On July 10th, 1943, British Eighth Army under General Montgomery and U.S. Seventh Army under General Patton landed on the south coast of Sicily. The Allies entered Messina on August 16th, the campaign having lasted 38 days.

[9] Henry Steele Commager, ed. *The Pocket History of the Second World War* (New York: Pocket Books, 1945), 334-335.

[10] The main attack on was planned for Salerno and Naples. Montgomery hoped to sweep north, to trap enemy forces between the toe and Salerno. *Ibid*, p. 337.

illustration, consider a northern front-line of 6 miles say, and an eastern front-line also of 6 miles.

The reader may now recall the square field of Sam the Farmer, from Part I of this paper. Algebraically, the calculations for change in the square area are the same in both examples. Following the clue from Section II.2 above on the advancement of a straight front-line, the focus now, however, is specifically on how the total length of the front-line fits into the picture.



Suppose, then, that the front-lines are both advanced by 1 mile, from 6 miles across to 7 miles across. Then the main territory gained would be along each front-line, with an extra 1 square mile at the north-east corner. So the total territory gained would be (6x1) + (6x1) + (1x1) square miles. If both front-lines are again advanced, this time from 7 miles across to 8 miles across, then the territory gained would be (7x1) + (7x1) + (1x1). And so on. That is, if the square region is x miles across the northern front-line and also x miles across the eastern front-line, then (except for the 1 square mile at the north-east corner), an advance of all front-lines by 1 mile gains

$2x$ square miles of territory. In other words, for every mile that the front-lines are advanced, the gain in square miles of territory is simply the length of the front lines that have been advanced.

But what if the front-lines along the x by x square territory are advanced some distance less than 1 mile? Say, for example, the front-lines are advanced only 1/2 mile.

Then the gain in territory would be (x)x(1/2) + (x)x(1/2) + (1/2)x(1/2) = ($2x$)x(1/2) +(1/4) square miles. As it turns out, we get essentially the same result as for the advance of 1 mile, that is, that the main contribution to the gain in territory is simply the length of the front-lines advanced (which is $2x$) times the distance advanced (which is 1/2). This time, however, with the advance of only 1/2 mile, the extra territory at the north-east corner is even less significant than before, for this time there is only an additional (1/2)x(1/2) square miles unaccounted for in the product (length of front-line)x(distance advanced).

As we did in Part I, to reach a basic pattern it can help to bring some algebra into play. For whatever the advance happens to be, h say, we can consider the ratio of square miles gained compared to distance advanced.

From the diagram, an advance of the front-lines by h provides a gain in territory equal to

$(x\times h)+(x\times h)+(h\times h)=2xh+h^2$. Note that when h is small, h^2 is even smaller. (Think of the example above with h = 1/2; and other examples like 1/3 of 1/3; 1/4 of 1/4, etc.) So the smaller h is, the closer the gain in territory is to being just $2xh$. To clinch things, let's calculate the ratio of territory gained compared to distance advanced. Dividing $2xh+h^2$ by h, we get $2x+h$.

What does this mean? In Part I, we already worked through to a notion of limit. What is new here? Remember that x is the initial distance across one edge of the square front-lines. Our question now adds further significance to the ratio by relating it to the total length of the advanced front-lines. Our result, then, is that the smaller the advance considered, the closer the ratio [territory gained]: [distance advanced] is to being exactly the total length of the front-lines, namely $2x$.

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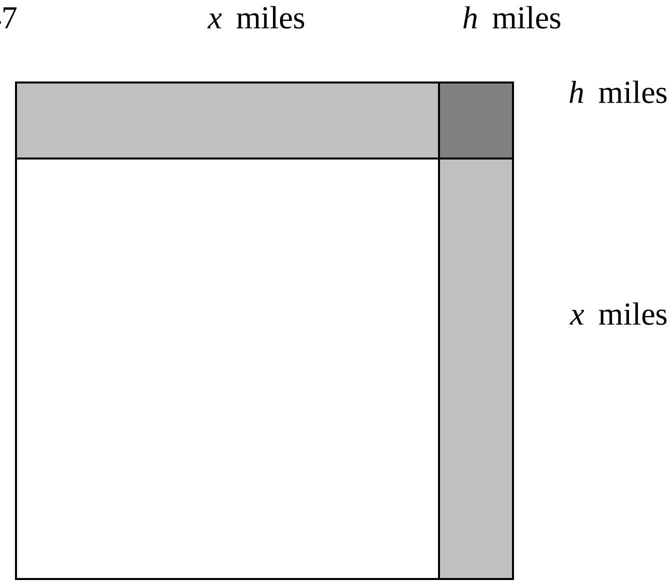
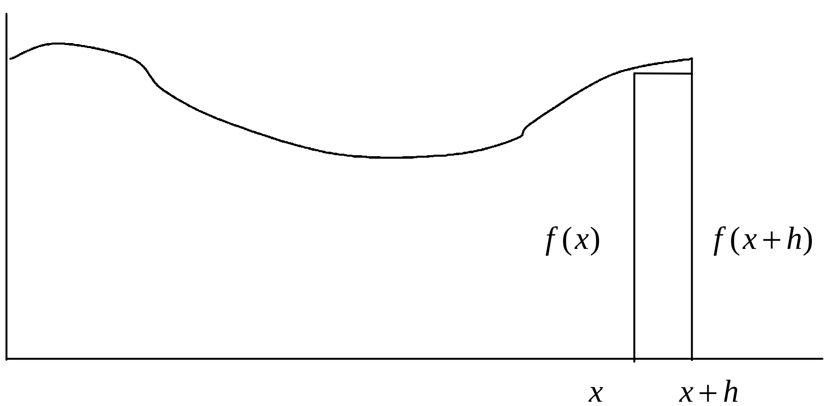


4. The Fundamental Theorem

You may realize that the calculations that we have done so far will work for other things besides advancing armies. Historically, x-y coordinate lines are added to the picture. So imagine a region ("territory") that is bounded on the left by the y-axis, across the bottom by the x-axis, across the top by a curve, and on the right by a vertical straight "front-line" at some x.



Just as the length of the front-line in a military campaign may vary as the army advances in the x-direction, the y-height of the region in the diagram also can vary. At an x along the x-axis, let's denote the height up the curve by $f(x)$. Let $A(x)$ denote the "area" of the region from the y-axis up to the front-line at x. If the front-line of the region is advanced from x to $x+h$, then the ratio of area gained compared to the advance h is $[A(x+h)-A(x)]/h$. But, when h is small, the main contribution to the area gained $A(x+h)-A(x)$ is the rectangular area $(f(x)\times h)$. So, for small h, except for the small corner region, the ratio is approximately $f(x)$, the length of the advancing front-line.[11] Or as it is said in modern Calculus (that is, after Cauchy, Bolzano and da Cunha), "the limit as h goes to zero of $[A(x+h)-A(x)]/h$ is $f(x)$". Using the notation of Leibniz, $\frac{dA}{dx}=f(x)$.

As mentioned in the *Introduction*, in the 19th century Riemann discovered a definition for *area* using limits (*limit* defined by Cauchy et al.) of finite sums (of rectangular approximations, that is, *Riemann Sums*). So in notation after Riemann, the area bounded on the left by the y-axis; below by the x-axis; above by the curve f; and on the right at x by the vertical ("front-line") of height $y=f(x)$, is denoted $A=\int_0^x f dx$. (The symbol $\int$ is a stylized S for "sum" - indicating the fact that the area is determined by a limit of sums.) Expressing our result in this notation, we get that the rate of change of area is $\frac{dA}{dx}=\frac{d}{dx}[\int_0^x f dx]=f(x)$. This result is called Part I of the Fundamental Theorem of Calculus.

Historically there were two problems: One problem was to

[11] The interested reader might now consult one of the standard textbooks, where this situation is discussed in the fuller context of a complete Calculus course. For instance, see James Stewart, *Calculus – 4th Edition* (Pacific Grove CA: Brooks/Cole, 1999), 338.

determine *relative rate of change* of a quantity (especially change of distance with respect to time); and the solution discovered by both Newton and Leibniz was given by the limit of ratios of differences -- now called the *derivative*. The other problem had a different focus, but was actually a special case. For the other problem was to determine (in particular) the rate of change of an advancing area.

Part I of the Fundamental Theorem relates these two problems, and at the same time provides a solution to what is often called the "inverse problem" or *anti-derivative* problem: Find a function whose derivative with respect to x (rate of change) is equal to a given a function $f(x)$. Part I of the Fundamental Theorem shows that any changing area whose front-line at x has length $f(x)$ will do. In particular, one such function will be $A = \int_0^x f dx$, representing the area under the graph of $f(x)$ itself.

Part II of the Fundamental Theorem concerns the solution of the anti-derivative problem. In specific content, however, Part II is a distinct result. For the basis of Part II of the theorem is not "area" or "Riemann integrals", as such, but regards the "uniqueness" of a solution to an anti-derivative problem. In particular, where Part I of the theorem produces one anti-derivative for a given function $f(x)$, we may enquire into the range of all possible anti-derivatives for $f(x)$. But, because a derivative is a *ratio of change*, and not an actual function value, it follows that if two functions $F(x)$ and $G(x)$ have the same derivative (that is, if they both change in exactly the same way), then as functions they must differ by a constant. (This is like saying that if two cars are driving along a road at the same speed in the same direction, then the distance between them stays the same.) Applying this to Part I, we find that not only is the area $A = \int_0^x f dx$ an anti-derivative for $f(x)$, but except for being able to add in a constant, it is the only anti-derivative for $f(x)$. So the general anti-derivative for $f(x)$ is given by

$F(x) = \int_0^x f dx + C$, where C can be any constant. This can be re-phrased using Riemann's notation: Let a and b be real numbers, and consider $F(b) - F(a) = \int_0^b f dx - \int_0^a f dx$. Under present hypotheses, this difference is a difference of areas under the graph of f, from $x = a$ to $x = b$; and in Riemann's notation this area is written $\int_a^b f dx$. The usual modern statement of Part II of The Fundamental Theorem is then that if F is any anti-derivative for f (that is $\frac{dF}{dx} = f$), then $F(b) - F(a) = \int_a^b f dx$.

Still keeping to the preliminary context (and so not requiring the precision of definitions and hypotheses on terms and functions involved), we can now state at least part of why The Fundamental Theorem is *fundamental*: (1) it shows exactly how to produce an anti-derivative; (2) it establishes the general form of all possible anti-derivatives; and (3) in empirical science, it provides a basis for the two-way pivot between possible abstract law involving a derivative and measurements of a coordinate that would be obtained in experiment.

Concluding Remarks: Calculus and Beyond

In this paper, which can be taken as a primer for a Calculus student, I have not discussed "velocity", "tangent lines" or "dynamics". Again, my purpose has not been an axiomatic development of the Calculus as such, but rather a fostering of the (two) basic insights that can help a student begin his or her reach toward the Calculus. Newton, of course, did much more than merely begin. And in his research into the dynamics of the planets, he made free use of coordinate techniques. Consequently, his results on rate of change apply also to limits of ratios obtained from slopes of secants of a graph. In other words, he discovered how to calculate the slope of a tangent line. His work then contributed toward the possibility of a verifiable geometry of space and time. In particular, from his (Calculus-based) abstract laws of gravitation, Newton was able to account for Kepler's Laws regarding the orbits of planets.[12]

As it later turned out, the truly fundamental nature of The Fundamental Theorem of Calculus allowed for a profound 20th century generalization that, because of its origins in the Calculus of Newton and Leibniz, also is called The Fundamental Theorem of Calculus (but, for "Manifolds"). The 20th century theorem also generalizes three 19th century integration theorems that were of special interest to physicists, namely Green's Theorem, The Divergence Theorem, and Stokes' Theorem. [So another name for the 20th century version of The Fundamental Theorem of Calculus (for Manifolds) is Stokes' Theorem (for Manifolds)]. The 20th century theorem distinguishes itself from the classical calculus theorem partly by the fact that it embraces higher dimensions.

There are clues to the generalization to higher dimensions

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[12] In Kepler's Three Laws of planetary motion, the first is a particular rule for orbits of planets; the second relates directly to a rate of change of an "area" (*area* not yet defined in Kepler's time); and the third directly regards certain space and time measurements. The three laws together are: (1) A planet's orbit is elliptical, with the sun at one of the foci; (2) The focal radius from the sun to the planet sweeps out equal areas in equal times; and (3) The squares of the times required for any two planets to make complete orbits about the sun is proportional to the cubes of their mean distances from the sun. (See, for example, David Burton's *The History of Mathematics: An Introduction, 4th ed* (New York: McGraw-Hill, 1999), 335. For a detailed and illuminating telling of Kepler's struggles and success, see Katz, Section 10.3.4.) Note that in Kepler's second law, the rate of change of the ("orbital") area is a constant for each planet. So while Kepler's Laws were subsumed by Newton's system of abstract laws, they remain important in their own right, as precursor to conservation laws and The Variational Calculus. Over the last century, conservation laws have been proving to be of central importance in empirical science. As shown by Noether et al., many conservation laws can be derived in Variational Calculus from the 20th century generalization of The Fundamental Theorem of Calculus for Manifolds. (See the last two paragraphs of this paper.)

in Calculus itself. Recall that $\frac{d}{dx}(x^2) = 2x$, $\frac{d}{dx}(x^3) = 3x^2$, $\frac{d}{dx}(x^4) = 4x^3$, and so on. If (as in the examples discussed in this paper) we consider the elementary geometry involved, then we get the following: The rate of change of a "2-dimensional square area" is the "1-dimensional length" of the advancing "front-line"; the rate of change of a "3-dimensional cubic volume" is the "2-dimensional area" of the advancing "front-surface"; and so on. It is a precise grasp and formulation of this "and so on" that leads to the generalized Fundamental Theorem of Calculus for Manifolds. For, going from the examples of powers of x, the rate of change of an n-dimensional quantity is the $n-1$- dimensional quantity of the advancing "front-surface". Where the classical theorem formulates the two-way pivot between measurement of a coordinate and abstract law involving a derivative, the generalized theorem formulates the two-way pivot between measurement of several coordinates and abstract law involving quantities with several rates of change. So the Fundamental Theorem of Calculus is not only fundamental in Calculus itself, but leads to fundamental results, in both mathematics and science, of on-going general significance. It is a well-named theorem.

Terrance J. Quinn teaches in the Department of Mathematics at Ohio University Southern. He can be reached at quinnt@ohio.edu.

Comments on this article can be sent to jmda@mun.ca.