Lattice Boltzmann Method for the Advection and Diffusion Equation in Shallow Water

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Abstract

Solute transport exists in most natural flows, which can be described by the advective-diffusive equation. In numerical modeling of such phenomenon, it is important to deal with irregular boundary and angular boundary points for hydrodynamic and water quality modeling. However, there is no clear control method for corner points in water quality modeling in existing literature. In this study, two-dimensional symmetrical flow in a cubic cavity with contaminant transport is investigated based on the presented control mechanisms: the lattice Boltzmann method for the advection and diffusion equation. For corner points of square cavity, a corner points controlling method proposed in this study shows great validity and reliability in the application of models.

Keywords: lattice Boltzmann method, advection and diffusion equation, shallow water, corner point

Introduction

The lattice Boltzmann method (LBM) has more than twenty years of development. During this development process, both theoretical and practical fields have been promoted and improved. With inherent parallel characteristics and highly effective processing of complex boundaries, LBM has achieved great success in solving numerical problems for multiple fields. It turns out that LBM has extensive application prospects.

In nature, there are many water bodies such as estuaries, lakes and bays, in which the depth is far less than the horizontal scale. In this scenario, shallow water equations can be utilized but as nonlinear partial differential equations, analytical solutions cannot be acquired directly. However, Zhou (2002a) and Zhou (2002b) established the lattice Boltzmann method for shallow water equations and simulated the turbulent flow through a sub-grid scale stress model. Linhao et al. (2005) preformed an analog on wind-driven marine circulation with LBM and obtained a fully explicit model through simplifying the collision operator by a second order approximate integral. Thömmes et al. (2007) applied LBM in a real fluid with complex terrain and irregular borders to verify the accuracy and efficiency of this method. Tubbs and Tsai (2009) proposed the multi-block lattice Boltzmann model and put it into solving three-dimensional wind-driven shallow water flow problems. Meanwhile, high performance CPU computing afforded implementation using Open Multiprocessing (Open MP). Two years later, Tubbs and Tsai (2011) applied GPU parallel computing to the LBM for shallow water.

In order to study the transition and diffusion of pollutants in shallow water, Li and Huang (2008) represented the coupling lattice Boltzmann model for convection and anisotropic dispersion. Ginzburg (2005) proposed a model with two relaxation times for advection and anisotropic-dispersion equations and gave special treatment to the Bhatager-Gross-Krook (BGK) model. Zhou (2011) put forward LBM for the anisotropic case on a rectangular lattice with a single relaxation time, which showed the accuracy and the stability of this model. Liu *et al*, (2010) and Liu *et al*, (2013) simulated transit flow and solute transport with coupling of the Boltzmann model and a large eddy simulation model in a local refined lattice.

Generally, the complicated boundaries in natural waters have strong effects on flow patterns. It is important for the shallow water models to treat boundary and corner points effectively and accurately. However, the existing studies lack appropriate handling of the corner points. Hence, the objective of this study is to simulate contaminant transport of jet-forced flow in a cubic cavity with processed angular boundary points.

In this research, the lattice Boltzmann method for the two-dimensional advection-diffusion equation with D2Q9 square lattices is applied to the symmetrical flow in a square cavity with injection. Then corner point conditions for hydrodynamic and water quality modeling in shallow water are briefly introduced.

Governing equations

Shallow water equations

Shallow water equations (Zhou, 2004) can be used to describe a variety of hydrodynamic process, which are reduced from the incompressible Navier–Stokes equations. The depth-averaged continuity and momentum equations can be written as

$$\frac{\partial h}{\partial t} + \frac{\partial h u_j}{\partial x_j} = 0$$

$$\frac{\partial (h u_i)}{\partial t} + \frac{\partial (h u_i u_j)}{\partial x_j} = -\frac{g \partial h^2}{2 \partial x_i} + \nu \frac{\partial^2 (h u_i)}{\partial x_j \partial x_j} + F_i$$

in which t is time; h is water depth; u is flow velocity in time and space; and F_i is the force term in direction i direction.

Advection and diffusion equations

Solute transport in shallow water can be described by the depth-averaged advection–diffusion equations, which are governed by

$$\frac{\partial (h C_i)}{\partial t} = h S_{ci} - \nabla \cdot (\mathbf{u} h C_i - \mathbf{K} \cdot \nabla (h C_i))$$

where C_i is the tracer or solute concentration; S_{ci} is the rate of production (sources) and destruction (sinks) of concentration; and $\mathbf{K}$ is the diffusion tensor.

Lattice Boltzmann Method

LABSWE

The two-dimensional shallow water equations with BGK collision operator for the lattice Boltzmann equation (LABSWE) is governed by

$$f_\alpha(\mathbf{x}+\mathbf{e}_\alpha\Delta t, t+\Delta t) - f_\alpha(\mathbf{x},t) = -\frac{1}{\tau}(f_\alpha - f_\alpha^{eq}) + \frac{\Delta t}{6e^2}e_{\alpha i}F_i$$

where f_α is the distribution function of particles; f_α^{eq} is the local equilibrium function; and τ is the relaxation time. The lattice speed is $e = \Delta x/\Delta t$; Δx is the grid cell size; and Δt is the time step.

In the case of two dimensions, the D2Q9 square lattice (shown in Figure 1) is more precise than the hexagonal lattice (Skordos, 1993) and furthermore, a square lattice is easier when handling boundary conditions. As a consequence, the D2Q9 square lattice is used in this hydrodynamic study.

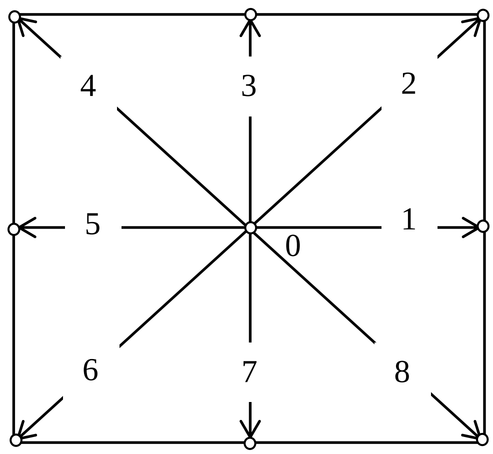

Figure 1. Lattice pattern: D2Q9 square lattice

For a D2Q9 lattice, the velocity vector of particle can be acquired by

$$e_\alpha = \begin{cases} (0,0), & \alpha = 0 \\ e\left[cos\frac{(\alpha-1)\pi}{4}, sin\frac{(\alpha-1)\pi}{4}\right], & \alpha = 1,3,5,7 \\ \sqrt{2}e\left[cos\frac{(\alpha-1)\pi}{4}, sin\frac{(\alpha-1)\pi}{4}\right], & \alpha = 2,4,6,8 \end{cases}$$

and the local equilibrium function in each direction reads

$$f_\alpha^{eq} = \begin{cases} h - \frac{5gh^2}{6e^2} - \frac{2h}{3e^2 u_i u_i}, & \alpha = 0 \\ \frac{gh^2}{6e^2} + \frac{h}{3e^2}e_{\alpha i}u_i + \frac{h}{2e^2}e_{\alpha i}e_{\alpha j}u_i u_j - \frac{h}{6e^2}u_i u_j, & \alpha = 1,3,5,7 \\ \frac{gh^2}{24e^2} + \frac{h}{12e^2}e_{\alpha i}u_i + \frac{h}{8e^2}e_{\alpha i}e_{\alpha j}u_i u_j - \frac{h}{24e^2}u_i u_j, & \alpha = 2,4,6,8 \end{cases}$$

By summing up the distribution function and ensuring conservation of mass and momentum, the physical variables h and u_i can be obtained by

$$h(\mathbf{x},t) = \sum_{\alpha} f_{\alpha}(\mathbf{x},t)$$
$$u_i(\mathbf{x},t) = \frac{1}{h(\mathbf{x},t)} \sum_{\alpha} e_{\alpha i} f_{\alpha}(\mathbf{x},t)$$

Lattice Boltzmann method for advection and diffusion equation

In this section, we use LBM for the two-dimensional advection-diffusion equation, which can be governed by

$$g_{\alpha}(\mathbf{x}+\mathbf{e}_{\alpha}\Delta t, t+\Delta t) - g_{\alpha}(\mathbf{x},t) = -\frac{1}{\tau_g}(g_{\alpha} - g_{\alpha}^{eq}) + \frac{S_c}{b}\Delta t$$

where g_{α} is the distribution function; g_{α}^{eq} is the local equilibrium distribution function; S_c is the source/sink term; and τ_g is the relaxation time. When a 9-speed rectangular grid is used, $b = 9$ (Zhou, 2011).

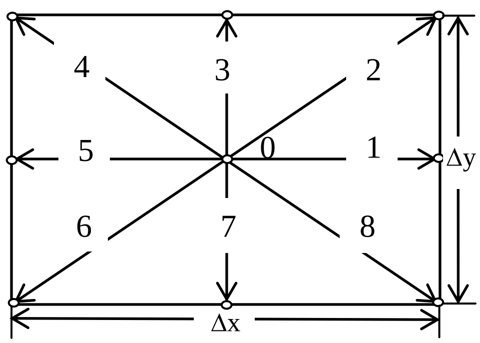

Figure 2. Lattice pattern: D2Q9 rectangular lattice

In the advection-diffusion module, 9-speed rectangular grid (Figure 2) can be used, which is more flexible and efficient than a square lattice owing to its different size in the x and y directions. The velocity vector of solute particle is

$$e_{g\alpha} = \begin{cases} (0,0), & \alpha = 0 \\ (\pm e_x, 0), (0, \pm e_y), & \alpha = 1,3,5,7 \\ (\pm e_x, \pm e_y), & \alpha = 2,4,6,8 \end{cases}$$

The equilibrium distribution functions g_{α}^{eq} defined by a dimensionless dispersion tensor λ_{ij} is

$$g_\alpha^{eq} = \begin{cases} (1 - \dfrac{\lambda_{yy} e_x^2 + \lambda_{xx} e_y^2}{e_x e_y}) hC, & \alpha = 0 \\ (\dfrac{1}{2}\dfrac{e_y}{e_x}\lambda_{xx} + \dfrac{u_x}{4e_{g\alpha x}}) hC, & \alpha = 1,5 \\ (\dfrac{1}{2}\dfrac{e_x}{e_y}\lambda_{yy} + \dfrac{u_y}{4e_{g\alpha y}}) hC, & \alpha = 3,7 \\ (\dfrac{1}{4}\dfrac{e_x e_y}{e_{g\alpha x} e_{g\alpha y}}\lambda_{xy} + \dfrac{u_i}{8e_{g\alpha i}}) hC, & \alpha = 2,4,6,8 \end{cases}$$

in which λ_{ij} can be expressed as

$$\lambda_{ij} = \frac{\mathbf{K}_{ij}}{\Delta t(\tau - 1/2) e_x e_y}$$

The concentration is defined in accordance with the distribution function as

$$C(\mathbf{x}, t) = \frac{1}{h} \sum_\alpha g_\alpha(\mathbf{x}, t)$$

The following three mathematical conditions are the properties of the local equilibrium distribution function,

$$\frac{1}{h} \sum_\alpha g_\alpha^{eq}(\mathbf{x}, t) = C(\mathbf{x}, t)$$

$$\frac{1}{h} \sum_\alpha e_{g\alpha i} g_\alpha^{eq}(\mathbf{x}, t) = u_i C(\mathbf{x}, t)$$

$$\frac{1}{h} \sum_\alpha e_{g\alpha i} e_{g\alpha j} g_\alpha^{eq}(\mathbf{x}, t) = \lambda_{ij} e_x e_y C(\mathbf{x}, t)$$

Numerical test

Two-dimensional symmetrical flow in cubic cavity with injection

Jet flow in a cubic basin is a classic example in shallow water problems. The eddy diffusivity coefficients are $\mathbf{K}_{xy} = 0$ and $\mathbf{K}_{xx} = \mathbf{K}_{yy} = 0.01$ m^2/s. The computational domain is a square cavity that is 1.5×1.5 m, and the width of the inflow and outflow channel as 0.156 m. The authors used 250×200 square lattices each with sides equal to 0.012 m and applied the no-slip boundary conditions to side walls. The velocity of $u = 0.1$ m/s and $v = 0$ m/s were imposed at the inflow and the water lever at the exit was fixed at 0.1 m. The bottom friction is set to zero. In the calculations, $\Delta t = 0.006$ s, $\nu = 7.84 \times 10^{-4}$ m^2/s and $\tau = 3\nu\Delta t / c^2 + 1/2 = 0.6176$. At the entrance, the pollutant concentration is 100 m^3/s.

At the corner points, there are five unknown distribution functions. For the upper left corner, the distribution functions f_α (α=1,2,6,7,8) are defined as

$$f_1(x,y) = \frac{2h(x+1,y-1)u(x,y)}{3e} + f_5(x,y)$$

$$f_7(x,y) = f_3(x,y)$$

$$f_8(x,y) = \frac{h(x+1,y-1)u(x,y)}{6e} + f_4(x,y)$$

$$f_2(x,y) = \frac{1}{2}(h(x+1,y-1)-(f_1(x,y)+f_3(x,y)+f_4(x,y)+f_5(x,y)+f_7(x,y)+f_8(x,y)+f_9(x,y))$$

$$f_6(x,y) = f_2(x,y)$$

Similarly, for the lower left corner, the distribution functions f_α (α=1,2,3,4,8) can be described as follows

$$f_1(x,y) = \frac{2h(x+1,y+1)u(x,y)}{3e} + f_5(x,y)$$

$$f_3(x,y) = f_7(x,y)$$

$$f_2(x,y) = \frac{h(x+1,y+1)u(x,y)}{6e} + f_6(x,y)$$

$$f_4(x,y) = \frac{1}{2}(h(x+1,y+1)-(f_1(x,y)+f_2(x,y)+f_3(x,y)+f_5(x,y)+f_6(x,y)+f_7(x,y)+f_9(x,y))$$

$$f_8(x,y) = f_4(x,y)$$

With regard to the upper right corner, the distribution functions f_α (α=4,5,6,7,8) is expressed by

$$f_5(x,y) = -\frac{2h(x-1,y-1)u(x,y)}{3e} + f_1(x,y)$$

$$f_7(x,y) = f_3(x,y)$$

$$f_6(x,y) = -\frac{h(x-1,y-1)u(x,y)}{6e} + f_2(x,y)$$

$$f_4(x,y) = \frac{1}{2}(h(x-1,y-1)-(f_1(x,y)+f_2(x,y)+f_3(x,y)+f_5(x,y)+f_6(x,y)+f_7(x,y)+f_9(x,y))$$

$$f_8(x,y) = f_4(x,y)$$

Lastly, the distribution functions f_α (α=2,3,4,5,6) in the lower right corner are described by

$$f_5(x,y) = -\frac{2h(x-1,y+1)u(x,y)}{3e} + f_1(x,y)$$

$$f_3(x,y) = f_7(x,y)$$

$$f_4(x,y) = -\frac{h(x-1,y+1)u(x,y)}{6e} + f_8(x,y)$$

$$f_2(x,y) = \frac{1}{2}(h(x-1,y+1)-(f_1(x,y)+f_3(x,y)+f_4(x,y)+f_5(x,y)+f_7(x,y)+f_8(x,y)+f_9(x,y))$$

$$f_6(x,y) = f_2(x,y)$$

The above method for corner point control in hydrodynamic studies can play a similar role in water quality simulations.

Results

In this section, two-dimensional jet flow in a cubic basin is utilized to demonstrate the method is able to simulate a recirculation in shallow water flows and control the corner points effectively.

It is choose six points for observing concentration simulations in this model, which is typical and representative separately along the cavity and shaft showing the injection and outflow shafts. According to the historical record, the model reaches steady state after 12000 iterations (Figure 3). The streamlines are displayed in Figure 4, which shows little difference in the analog result obtained by Li *et al.* (2009). Owing to circulating flows on both sides of the central shaft are within the cubic cavity, we can recognize that in the circulating flows, area concentration is higher than in the center section as seen in Figure 5. Figure 6 describes the trends of concentration along the horizontal and vertical axes, which show no apparent errors arising from the treatment of corners in this study and agrees with the common understanding of solute transport in such flow patterns.

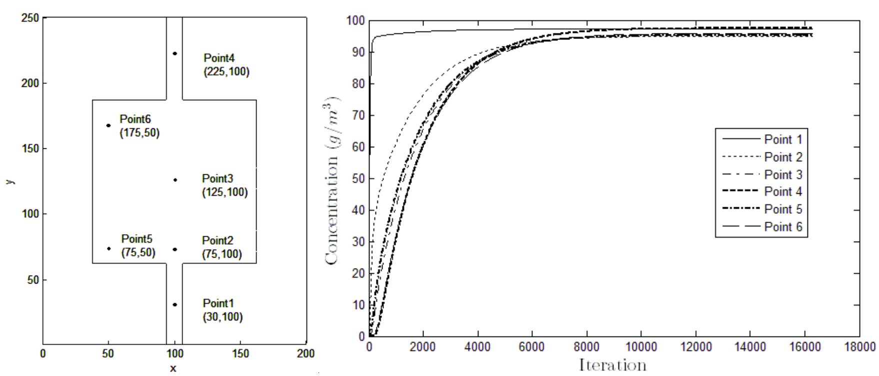

Figure 3. The position (left) and historical record (right) of six points

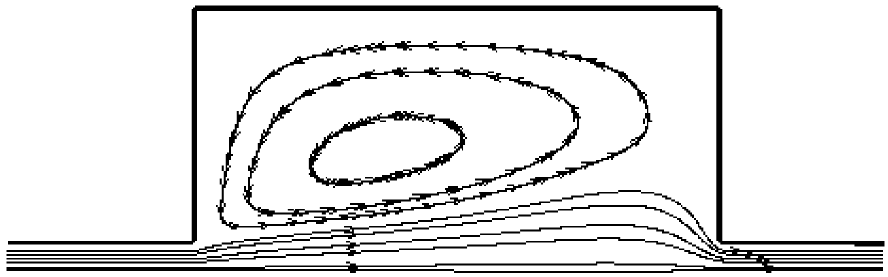

Figure 4. Streamlines in upper part of basin

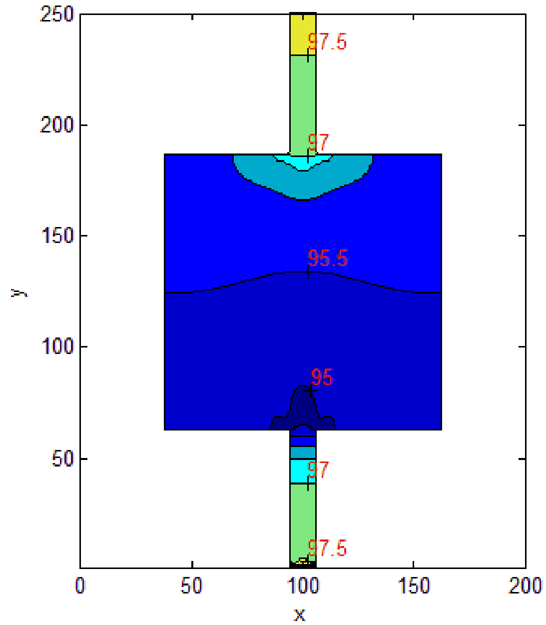

Figure 5. Contours of concentration distribution (g/m^3)

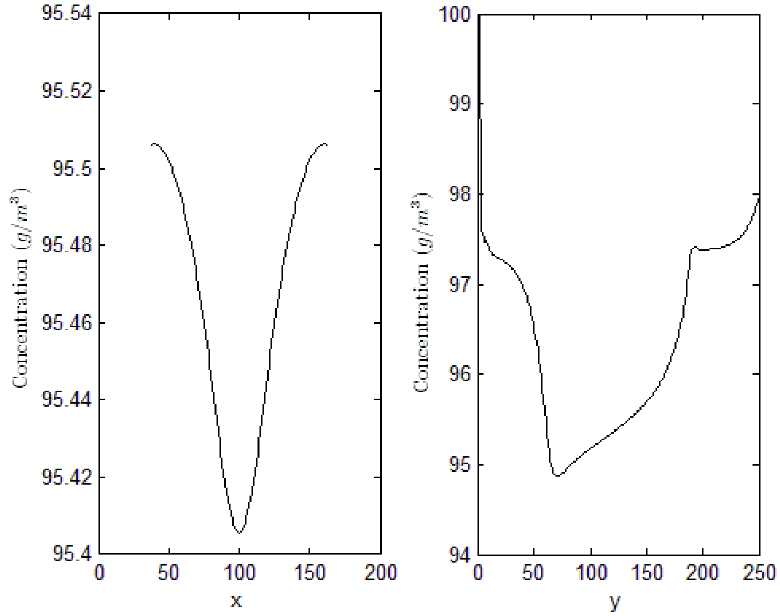

Figure 6. Horizontal (left) and vertical (right) changes in concentration

Conclusions

In this study, a lattice Boltzmann model for the convective-diffusive equation was applied to a two-dimensional symmetrical jet flow in a square cavity with a D2Q9 lattice pattern. In addition, a method for corner point control is proposed in hydrodynamic modelling that may be suitable for water quality modeling. The results demonstrate that the proposed method is effective and has good applicability, which may help to eliminate calculation errors at corner points.

Acknowledgement

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References

Ginzburg I. (2005). Equilibrium-type and link-type lattice Boltzmann models for generic advection and anisotropic-dispersion equation. *Advances in Water resources*, *28*(11), 1171-1195.

Li Y and Huang P. (2008). A coupled lattice Boltzmann model for advection and anisotropic dispersion problem in shallow water. *Advances in Water Resources*, 31(12): 1719-1730.

Li Y., Duan Y. L., Guo Y. and Liu R. X. (2009). Numerical simulation of laminar jet-forced flow using lattice Boltzmann method. *Applied Mathematics and Mechanics*, *30*, 445-453.

Linhao Z., Shide F. and Shouting, G. (2005). Wind-driven ocean circulation in shallow water lattice Boltzmann model. *Advances in Atmospheric Sciences*, *22*(3), 349-358.

Liu H., Zhou J. G. and Burrows, R. (2010). Lattice Boltzmann simulations of the transient shallow water flows. *Advances in Water Resources*, *33*(4), 387-396.

Liu H., Zhou J. G. Li, M. and Zhao, Y. (2013). Multi-block lattice Boltzmann simulations of solute transport in shallow water flows. *Advances in Water Resources*, *58*, 24-40.

Skordos P. A. (1993). Initial and boundary conditions for the lattice Boltzmann method. *Physical Review E*, *48*(6), 4823-4842.

Thömmes G., Seaïd M. and Banda, M. K. (2007). Lattice Boltzmann methods for shallow water flow applications. *International Journal for Numerical Methods in Fluids*, *55*(7), 673-692.

Tubbs K. R. and Tsai F. T. C. (2009). Multilayer shallow water flow using lattice Boltzmann method with high performance computing. *Advances in Water Resources*, *32*(12), 1767-1776.

Tubbs K. R. and Tsai F. T. C. (2011). GPU accelerated lattice Boltzmann model for shallow water flow and mass transport. *International Journal for Numerical Methods in Engineering*, *86*(3), 316-334.

Zhou J. G. (2002a). A lattice Boltzmann model for the shallow water equations. *Computer methods in applied mechanics and engineering*, *191*(32), 3527-3539.

Zhou J. G. (2002b). A lattice Boltzmann model for the shallow water equations with turbulence modelling. *International Journal of Modern Physics C*, *13*(08), 1135-1150.

Zhou J. G. (2004). Lattice Boltzmann methods for shallow water flows. Vol. 4. Berlin: Springer.

Zhou J. G. (2011). Lattice Boltzmann method for advection and anisotropic dispersion equation. *Journal of Applied Mechanics*, 78(2).